Dr M. S. Meyer

3 - 06

THE

ELEMENTS

OF

ALGEBRA,

DESIGNED FOR BEGINNERS.

BY ELIAS LOOMIS, LL.D.,

PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN THE UNIVERSITY OF THE CITY OF NEW YORK, AUTHOR OF A "COURSE OF MATHEMATICS," "RECENT PROGRESS OF ASTRONOMY," ETC., ETC.

SIXTH EDITION.

NEW YORK:

HARPER & BROTHERS, PUBLISHERS,

329 & 331 PEARL STREET,

FRANKLIN SQUARE.

1856.

PREFACE.

Having understood that my "Treatise on Algebra," which was designed primarily for the use of colleges, has been introduced into many academies and high schools, and employed in the instruction of classes younger than those for whom it was originally prepared, I have thought that a more elementary work, expressly designed for beginners, might be favorably received. The present volume was intended for the use of students who have just completed the study of Arithmetic; and it is believed that any person, however young, who has acquired a tolerably thorough knowledge of the principles of that science, may proceed at once to this volume with pleasure and profit. I have endeavored to render the transition from Arithmetic to Algebra both easy and natural. This I have done by applying the algebraic symbols to problems so simple that they might be readily solved by the principles of Arithmetic alone. Having conducted the student through a considerable series of simple problems, I proceed, by easy steps, to develop some of them in a more general form. The student is thus led to represent known as well as unknown quantities by letters of the alphabet, and perceives the necessity of establishing rules for performing the various operations of addition, subtraction, multiplication, and division upon quantities thus represented. It is believed that the beginner will study these abstract principles with more satisfaction, than if he had been allowed no previous exercise on problems which indicate their importance.

I have omitted from this volume all such topics as it was supposed would occasion any serious embarrassment to the young learner, and which were not essential to the clear comprehension of the topics actually introduced. It is hoped that the book will be found sufficiently clear and simple to be adapted to the wants of a large class of students in our common schools.

The study of Algebra may properly be commenced at an early stage of education. As soon as the mind has acquired some degree of maturity, and has become familiar with the principles of common Arithmetic, it is prepared to understand the elementary principles of Algebra. This study is admirably adapted to strengthen the reasoning faculties, to lead the mind to rely upon its own resources, and to cultivate those habits of independent thinking which are alike important to the scholar and to the man of business

CONTENTS.

SECTION I.

PRELIMINARY DEFINITIONS AND FIRST PRINCIPLES.

SECTION II.

ADDITION.

SECTION III.

SUBTRACTION.

SECTION IV.

MULTIPLICATION.

SECTION V.

DIVISION.

SECTION VI.

FRACTIONS.

SECTION VII.

EQUATIONS OF THE FIRST DEGREE.

SECTION VIII.

EQUATIONS WITH TWO UNKNOWN QUANTITIES.

SECTION IX.

EQUATIONS WITH THREE OR MORE UNKNOWN QUANTITIES.

SECTION X.

INVOLUTION AND POWERS.

SECTION XI.

EVOLUTION AND RADICAL QUANTITIES.

ELEMENTS OF ALGEBRA.

SECTION I.

PRELIMINARY DEFINITIONS AND FIRST PRINCIPLES.

(*Article* 1.) ARITHMETIC is the art or science of numbering. It treats of the nature and properties of numbers, but it is limited to certain methods of calculation which occur in common practice.

(2.) Algebra is a branch of mathematics which enables us to abridge and generalize the reasoning employed in the solution of all questions relating to numbers. It has been called by Newton *Universal Arithmetic.*

(3.) One advantage which Algebra has over Arithmetic arises from the introduction of symbols, by which the operations to be performed are readily indicated to the eye.

(4.) The following are some of the more common symbols employed in Algebra.

The sign + (an erect cross) is named *plus*, and is employed to denote the *addition* of two or more numbers. Thus, 5+3 signifies that we must add 3 to the number 5, in which case the result is 8. In the same manner, 5+7 is equal to 12; 11+8 is equal to 19, etc.

QUESTIONS.—What is Arithmetic? What is Algebra? What advantage has Algebra over Arithmetic? What does the sign plus denote?

We also make use of the same sign to connect several numbers together. Thus 7+5+3 signifies that to the number 7 we must add 5 and also 3, which make 15.

So, also, the sum of 6+5+11+9+2+8 is equal to 41.

(5.) The sign − (a horizontal line) is called *minus*, and indicates that one quantity is to be *subtracted* from another. Thus, 7−4 signifies that the number 4 is to be taken from the number 7, which leaves a remainder of 3. In like manner, 11−6 is equal to 5, and 16−10 is equal to 6, etc.

Sometimes we may have several numbers to subtract from a single one. Thus, 16−5−4 signifies that 5 is to be subtracted from 16, and this remainder is to be further diminished by 4, leaving 7 for the result. In the same manner, 30−8−6−2−5 is equal to 9.

(6.) The sign × (an inclined cross) is employed to denote the *multiplication* of two or more numbers. Thus, 5×3 signifies that 5 is to be multiplied by 3, making 15.

(7.) The character ÷ (a horizontal line with a point above and another below it) shows that the quantity which precedes it is to be divided by that which follows.

Thus, 24÷6 signifies that 24 is to be divided by 6, making four.

Generally, however, the division of two numbers is indicated by writing the dividend above the divisor and drawing a line between them.

QUEST.—What does the sign minus indicate? What is the sign of multiplication? How may division be denoted?

Thus, instead of $24 \div 6$, we usually write $\frac{24}{6}$.

(8.) The sign = (two horizontal lines), when placed between two quantities, denotes that they are equal to each other.

Thus, $7+6=13$ signifies that the sum of 7 and 6 is equal to 13. So, also, \$1=100 cents, is read one dollar equals one hundred cents; 3 shillings=36 pence, is read three shillings are equal to thirty-six pence.

(9.) The following examples will afford an exercise upon the preceding symbols.

EXAMPLE 1. $5\times8+12-4=6\times9-\frac{12}{3}-2$.

This may be read as follows: The product of 5 by 8, increased by 12 and diminished by 4, is equal to 6 times 9, diminished by one third of 12, and again diminished by 2.

To find the value of each side of this equation, we multiply 5 by 8, which gives 40; adding 12 to this product, we obtain 52, and, subtracting 4, we have 48. Again, the product of 6 by 9 is 54, which, diminished by one third of 12, leaves 50, and, subtracting 2 from this result, we have 48, as before.

Verify the following examples in the same manner:

Ex. 2. $7\times9-5+14=8\times6+20+\frac{20}{5}$.

Ex. 3. $12+\frac{28}{4}-3=42-17-9$.

Ex. 4. $3+25-7=\frac{42}{6}-5+19$.

QUERY.—What is the sign of equality?

(10.) When a problem is proposed for solution, it is generally required to find one or more quantities which are unknown. It is convenient to have signs to represent these unknown quantities, so that all the operations which are required to be performed may be presented at once in a single view.

The signs generally employed to represent these unknown quantities are some of the last letters of the alphabet; as, x, y, z, etc.

These principles will be best understood after attending to a few practical examples.

Problem 1. A boy bought an apple and an orange for 6 cents; for the orange he gave twice as much as for the apple. How much did he give for each?

Let x represent the number of cents he gave for the apple, then $2x$ will represent the number of cents he gave for the orange. Now these, added together, must make the sum given for both, which was 6 cents; that is,

$$x+2x=6.$$

But twice x, added to once x, makes three times x; that is, $3x=6$;

and if three times x is equal to 6, once x must be equal to 2; that is,

$$x=\frac{6}{3}=2.$$

Therefore the apple cost 2 cents, and the orange 4 cents, the sum of which is 6 cents, according to the conditions of the problem.

Prob. 2. A man having a horse and cow, was asked what was the value of each. He answered that the horse was worth three times as much as the cow, and

QUEST.—How are unknown quantities represented?

together they were worth 60 dollars. What was the value of each?

Let x represent the number of dollars equal to the value of the cow, then $3x$ will represent the value of the horse. These, added together, must make 60, according to the conditions of the problem; that is,

$$x+3x=60.$$

But three times x, added to once x, makes four times x; that is,

$$4x=60;$$

and if four times x is equal to 60, once x must be equal to 15; that is,

$$x=\frac{60}{4}=15.$$

Therefore the cow was worth 15 dollars, and the horse, being worth three times as much as the cow, amounted to 45 dollars. The sum of 15 and 45 is 60, according to the conditions of the problem.

Prob. 3. Said Charles to Thomas, my purse and money together are worth 10 dollars, but the money is worth four times as much as the purse. How much money was there in the purse, and what was the value of the purse?

Let x represent the value of the purse.

Then $4x$ will represent the value of the money it contained. Then, by the problem, we must have

$$x+4x=10,$$

or,

$$5x=10$$

Hence,

$$x=\frac{10}{5}=2.$$

Therefore the purse was worth 2 dollars, and the money 8 dollars, the sum of which is 10 dollars.

Prob. 4. Two men, A and B, trade in company. B puts in five times as much money as A. They gain 660 dollars. What is each man's share of the gain?

Let x represent A's share.

Then $5x$ will represent B's share.

Hence, $x+5x=660$,

or, $6x=660$.

Hence, $x=\frac{660}{6}$, or 110.

Therefore A's share is 110 dollars, and B's share is 550 dollars, the sum of which is 660 dollars.

Prob. 5. A gentleman, meeting two poor persons, divided 21 shillings between them, giving to the second six times as much as to the first. How much did he give to each?

Let $x=$ the shillings he gave to the first.

Then $6x=$ the shillings he gave to the second.

Therefore, $x+6x=21$,

or, $7x=21$.

Hence, $x=\frac{21}{7}$, or 3.

Therefore he gave 3 shillings to the first and 18 shillings to the second, the sum of which is 21 shillings.

Prob. 6. A gentleman bequeathed 144 pounds to two servants upon condition that one should receive seven times as much as the other. How much did each receive?

Let $x=$ the smallest share.

Then $7x=$ the share of the other.

Therefore, $x+7x=144$,

or, $8x=144$.

Hence, $x=\frac{144}{8}$, or 18.

Therefore one received 18 pounds, and the other 126, the sum of which is 144 pounds.

Prob. 7. A draper bought two pieces of cloth, which together measured 171 yards. The second piece contained eight times as many yards as the first. What was the length of each?

Let $x=$ the number of yards in the first piece.

Then $8x=$ the number of yards in the second piece.

Therefore, $x+8x=171$,

or, $9x=171$.

Hence, $x=\frac{171}{9}$, or 19.

Therefore the length of the first piece was 19 yards, and that of the second was 152 yards, the sum of which is 171 yards.

Prob. 8. A man being asked the price of his horse, answered that his horse and saddle together were worth 90 dollars, but the horse was worth nine times as much as the saddle. What was each worth?

Let $x=$ the price of the saddle.

Then $9x=$ the price of the horse.

Therefore, $x+9x=90$,

or, $10x=90$.

Hence, $x=\frac{90}{10}$, or 9.

Therefore the saddle was worth 9 dollars, and the horse was worth 81 dollars, the sum of which is 90 dollars

Prob. 9. A cask which held 143 gallons was filled with a mixture of brandy and water, and there was ten times as much brandy as water. How much was there of each?

Let $x=$ the gallons of water.

Then $10x=$ the gallons of brandy.

Therefore, $x+10x=143$,

or, $11x=143$.

Hence, $x=\frac{143}{11}$, or 13.

Therefore there were 13 gallons of water and 130 gallons of brandy, the sum of which is 143 gallons.

(11.) The pupil will observe that when a problem is proposed for solution, the first thing to be done is to find an expression which shall contain the unknown quantity, and which shall be equal to a given quantity. Then, from this expression, by arithmetical operations, we deduce the value of the unknown quantity.

This expression of equality between two quantities is called an *equation*. Thus, $x+10x=143$, is an equation.

The quantity or quantities on the left side of the sign of equality are called the *first member* of the equation; those on the right, the *second member* of the equation.

Thus, $x+10x$ is the first member of the above equation, and 143 is the second member.

(12.) Quantities connected by the signs + and − are called *terms*. Thus, x and $10x$ are terms in the above equation.

A number written before a letter, showing how many

QUEST.—What is the course pursued in solving a problem? What is an equation? What are the members of an equation? What are terrtimes the letter is to be taken, is called the *coefficient* of that letter. Thus, in the quantity $10x$, 10, is the coefficient of x.

(13.) The solution of a problem, by Algebra, consists of two distinct parts:

1. To express the conditions of the problem algebraically; that is, to *form* the equation.

2. To find the value of the unknown quantity after the equation is formed; that is, to *reduce* the equation.

(14.) It is impossible to give a general rule which will enable us to translate every problem into algebraic language. This must be learned by practice But rules may be given for reducing the equation after it is formed.

After the preceding problems were reduced to equations, *the first step was to reduce all the terms containing the unknown quantity to a single term*, which was done by adding the coefficients. *The second step was to divide each member of the equation by the coefficient of the unknown quantity.*

In a similar manner may the following equations be solved.

Prob. 10. A gentleman, having 36 shillings to divide between a man and a boy, wishes to give to the man twice as much as to the boy. How much must he give to each.

Ans. 12 shillings to the boy,
and 24 shillings to the man.

(15.) All examples of this kind admit of proof. The results are proved to be correct when they fulfill all

QUEST.—What is a coefficient? How is an algebraic problem solved? How is an equation reduced?

the conditions of the problem. In the preceding problem there are two conditions; first, that the boy and man together receive 36 shillings; second, that the man receives twice as much as the boy. The numbers 12 and 24 fulfill both of these conditions. All the results in the following problems should be verified in a similar manner.

Prob. 11. A man having a horse and cow, was asked what was the value of each. He answered that the horse was worth three times as much as the cow, and together they were worth 72 dollars. What was the value of each? *Ans.* The cow was worth 18 dollars,
and the horse 54 dollars.

Prob. 12. Said Thomas to Charles, my purse and money together are worth 15 dollars, but the money is worth four times as much as the purse. How much money was there in the purse, and what was the value of the purse?

Ans. The purse was worth 3 dollars,
and the money was worth 12 dollars.

Prob. 13. Two men, A and B, trade in company; but B puts in five times as much money as A. They gain 900 dollars. What is each man's share of the gain? *Ans.* A's share is 150 dollars,
and B's share is 750 dollars.

Prob. 14. A gentleman, meeting two poor persons, divided 28 shillings between them, giving to the second six times as much as to the first. How much did he give to each?

Ans. He gave 4 shillings to the first,
and 24 shillings to the second.

QUEST.—How may the answers obtained be verified?

Prob. 15. A gentleman bequeathed 200 dollars to two servants upon condition that one should receive seven times as much as the other. How much did each receive? *Ans.* One received 25 dollars,
and the other 175 dollars.

Prob. 16. A draper bought two pieces of cloth, which together measured 144 yards. The second piece contained eight times as many yards as the first. What was the length of each?

Ans. The first piece contained 16 yards,
and the second 128 yards.

Prob. 17. A man being asked the price of his horse, answered that his horse and saddle together were worth 120 dollars, but the horse was worth nine times as much as the saddle. What was each worth?

Ans. The saddle was worth 12 dollars,
and the horse 108 dollars.

Prob. 18. A cask which held 132 gallons was filled with a mixture of brandy and water, and there was ten times as much brandy as water. How much was there of each?

Ans. There were 12 gallons of water,
and 120 gallons of brandy.

The following problems are similar to the preceding, except that an additional term is introduced.

Prob. 19. A gentleman, meeting three poor persons, divided 72 cents among them; to the second he gave twice, and to the third three times as much as to the first. What did he give to each?

Let $x=$ the sum given to the first.

Then $2x=$ the sum given to the second,

and $3x=$ the sum given to the third.

Then, by the conditions of the problem,

$$x+2x+3x=72.$$

That is, $6x=72$,

or, $x=12$, the sum given to the first.

Therefore he gave 24 cents to the second, and 36 cents to the third.

The learner should verify this, and all the subsequent results.

Prob. 20. Three men, A, B, and C, found a purse of money containing 119 dollars, but not agreeing about the division of it, each took as much as he could get. A got a certain sum, B got twice as much as A, and C four times as much as A. How many dollars did each get?

Let $x=$ the number of dollars A got.

Then $2x=$ the number of dollars B got,

and $4x=$ the number C got.

These, added together, must make 119 dollars, the whole sum to be divided. Hence,

$$x+2x+4x=119.$$

Uniting all the x's,

$$7x=119.$$

Hence, $x=17=$A's share;

$2x=34=$B's share;

$4x=68=$C's share.

Prob. 21. Three men, A, B, and C, trade in company. A puts in a certain sum, B puts in three times as much as A, and C puts in five times as much as A. They gain 657 dollars. What is each man's share of the gain?

Let $x=$ A's share.

Then $3x=$ B's share,
and $5x=$ C's share.

Hence, $x+3x+5x=657$,
or, $9x=657$.

Hence, $x=73$ dollars $=$ A's share;
$3x=219$ dollars $=$ B's share;
$5x=365$ dollars $=$ C's share.

Prob. 22. A gentleman left 15,000 dollars to be divided between his widow, his son, and daughter. He directed that his son should receive three times as much as his daughter, and his widow six times as much as his daughter. Required the share of each.

Let x represent the share of his daughter.
Then $3x$ will represent the share of his son,
and $6x$ will represent the share of his widow.

Hence, $x+3x+6x=15,000$,
or, $10x=15,000$.

Hence, $x=1500$ dollars $=$ the daughter's share;
$3x=4500$ dollars $=$ the son's share;
and $6x=9000$ dollars $=$ the widow's share.

Prob. 23. A farmer bought some oxen, some cows, and some sheep. The number of them all together was 48. There were three times as many cows as oxen, and four times as many sheep as oxen. How many were there of each sort?

Let x denote the number of oxen.
Then $3x$ will denote the number of cows,
and $4x$ will denote the number of sheep.

Hence, $x+3x+4x=48$,
or, $8x=48$.

Hence, $x=6$, the number of oxen;
$3x=18$, the number of cows;
and $4x=24$, the number of sheep.

Prob. 24. A boy bought some oranges, some lemons, and some pears. The number of them all together was 132. There were four times as many lemons as oranges, and six times as many pears as oranges. How many were there of each?

Let $x=$ the number of oranges.

Then $4x=$ the number of lemons,
and $6x=$ the number of pears.

Hence, $x+4x+6x=132$,
or, $11x=132$.
Hence, $x=12$, the number of oranges;
$4x=48$, the number of lemons;
and $6x=72$, the number of pears.

Prob. 25. Three persons are to share 364 dollars in the following manner. The second is to have five times as much as the first, and the third seven times as much as the first. What is the share of each?

Let $x=$ the share of the first.

Then $5x=$ the share of the second,
and $7x=$ the share of the third.

Therefore, $x+5x+7x=364$,
or, $13x=364$.
Hence, $x=28$, the share of the first;
$5x=140$, the share of the second;
$7x=196$, the share of the third.

Prob. 26. A draper bought three pieces of cloth, which together measured 90 yards. The second piece was six times as long as the first, and the third was eight times as long as the first. What was the length of each?

Let $x=$ the length of the first piece.

Then $6x=$ the length of the second piece.

and $8x=$ the length of the third piece.

Therefore, $x+6x+8x=90$,

or, $15x=90$.

Hence, $x=$ 6 yards, the length of the first piece;

$6x=36$ yards, the length of the second piece;

and $8x=48$ yards, the length of the third piece.

Prob. 27. A cask, which held 135 gallons, was filled with a mixture of brandy, wine, and water. It contained five times as much wine as water, and nine times as much brandy as water. What quantity was there of each?

Let $x=$ the number of gallons of water.

Then $5x=$ the number of gallons of wine.

and $9x=$ the number of gallons of brandy.

Therefore, $x+5x+9x=135$,

or, $15x=135$.

Hence, $x=$ 9, the gallons of water;

$5x=45$, the wine;

and $9x=81$, the brandy.

Prob. 28. A gentleman, meeting three poor persons, divided 90 cents among them; to the second he gave twice, and to the third three times as much as to the first. What did he give to each?

Ans. He gave 15 cents to the first,
30 cents to the second,
45 cents to the third.

Prob. 29. Three men, A, B, and C, found a purse of money containing 175 dollars, but not agreeing

about the division of it, each took as much as he could get. A got a certain sum, B got twice as much as A, and C four times as much as A. How many dollars did each get? *Ans.* A got 25 dollars,
B got 50 dollars,
C got 100 dollars.

Prob. 30. Three men, A, B, and C, trade in company. A puts in a certain sum, B puts in three times as much as A, and C puts in five times as much as A. They gain 765 dollars. What is each man's share of the gain?

Ans. A's share is 85 dollars;
B's share is 255 dollars;
C's share is 425 dollars.

Prob. 31. A gentleman left 24,000 dollars to be divided between his widow, his son, and his daughter. He directed that his son should receive three times as much as his daughter, and his widow six times as much as his daughter. Required the share of each

Ans. The daughter's share was 2,400 dollars;
the son's share was 7,200 dollars;
the widow's share was 14,400 dollars.

Prob. 32. A farmer bought some oxen, some cows, and some sheep. The number of them all together was 64. There were three times as many cows as oxen, and four times as many sheep as oxen. How many were there of each sort?

Ans. There were 8 oxen,
24 cows,
32 sheep.

Prob. 33. A boy bought some oranges, some lemons, and some pears. The number of them all together was 165. There were four times as many lemons as oranges, and six times as many pears as oranges. How many were there of each?

Ans. There were 15 oranges,
60 lemons,
90 pears.

Prob. 34. Three persons are to share 598 dollars in the following manner. The second is to have five times as much as the first, and the third seven times as much as the first. What is the share of each?

Ans. The share of the first is 46 dollars;
the second is 230 dollars;
the third is 322 dollars.

Prob. 35. A draper bought three pieces of cloth, which together measured 105 yards. The second piece was six times as long as the first, and the third was eight times as long as the first. What was the length of each?

Ans. There were 7 yards of the first piece,
42 yards of the second piece,
56 yards of the third piece.

Prob. 36. A cask which held 120 gallons was filled with a mixture of brandy, wine, and water. It contained five times as much wine as water, and nine times as much brandy as water. What quantity was there of each?

Ans. There were 8 gallons of water,
40 gallons of wine,
72 gallons of brandy.

The following problems are similar to the preceding, except that a new term has been introduced. It is recommended to the pupil that he should endeavor to solve these problems unaided, before reading the solutions here given; and after he has succeeded, let him compare his solution with that of the book.

Prob. 37. The number of days that four workmen were employed were severally as the numbers 1, 2, 3, 4; and the number of days' work performed by them all was 170. How many days was each workman employed?

Let x represent the number of days the first was employed.

Then $2x$ will represent the days the second was employed,

$3x$ the third,

and $4x$ the fourth.

Then, by the conditions of the problem,

$$x+2x+3x+4x=170;$$

that is, $10x=170$,

or, $x=17$ days the first was employed;

$2x=34$, the second;

$3x=51$, the third;

$4x=68$, the fourth.

The sum of all the numbers is 170.

Prob. 38. The estate of a bankrupt, valued at 14,400 dollars, is to be divided among four creditors according to their respective claims. The debts due to B are double those due to A; those due to C are four times those due to A; and those due to D are five times those due to A. What sum must each receive?

Let $x=$ the sum A receives;

then $2x=$ the sum B receives;

$4x=$ the sum C receives;

$5x=$ the sum D receives.

Therefore, $x+2x+4x+5x=14,400$;
that is, $12x=14,400$,
or, $x=1200$,
from which the shares of the other three are easily obtained.

Prob. 39. A draper has four pieces of cloth whose united value is 224 dollars. The value of the second piece is double that of the first; the value of the third is five times that of the first; and the value of the fourth is six times that of the first. What is the value of each?

Let x represent the value of the first;
then $2x$ will represent the value of the second,
$5x$ the third,
$6x$ the fourth.

Hence, $x+2x+5x+6x=224$,
that is, $14x=224$,
or, $x=16$, the value of the first piece,
from which the value of the other pieces is readily obtained.

Prob. 40. A grocer has four casks, which together will contain 144 gallons. The capacity of the second is twice that of the first; the capacity of the third is six times that of the first; and the capacity of the fourth is seven times that of the first. What is the capacity of each?

Let $x=$ the capacity of the first cask;
then $2x=$ the capacity of the second,
$6x$ the third,
and $7x$ the fourth.

Therefore, $x+2x+6x+7x=144$;
that is, $16x=144$,

or, $x = 9,$

$2x =$, etc.

Prob. 41. Four persons purchased a farm in company for 4680 dollars, of which B paid three times as much as A, C paid four times as much as A, and D paid five times as much as A. What did each pay?

Let $x =$ the sum A paid;

$3x =$ the sum B paid;

$4x =$ the sum C paid;

$5x =$ the sum D paid.

Therefore, $x+3x+4x+5x=4680$;

that is, $13x=4680,$

or, $x= 360,$

$3x =$, etc.

Prob. 42. Four persons, A, B, C, and D, drew prizes in a lottery to the amount of 2250 dollars. B drew three times as much as A, C drew five times as much as A, and D drew six times as much as A. What did each person draw?

Let $x =$ the sum A drew;

then $3x =$ the sum B drew;

$5x =$ the sum C drew;

$6x =$ the sum D drew.

Therefore, $x+3x+5x+6x=2250$;

that is, $15x=2250,$

or, $x= 150,$

$3x =$, etc.

Prob. 43. An estate of 5440 dollars is to be divided between four heirs, A, B, C, and D. B is to receive three times as much as A, C is to receive six times as much as A, and D is to receive seven times as much as A. How much did each receive?

Let $x=$ A's share;
$3x=$ B's share;
$6x=$ C's share;
$7x=$ D's share.

Therefore, $x+3x+6x+7x=5440$;
that is, $17x=5440$,
or, $x=320$,
$3x=$, etc.

Prob. 44. A farmer has 144 fruit-trees in his orchard, consisting of plum, cherry, peach, and apple-trees. The number of cherry-trees is four times the number of plum-trees; the number of peach-trees is five times the number of plum-trees; and the number of apple-trees is six times the number of plum-trees. What is the number of each sort?

Let $x=$ the number of plum-trees;
$4x=$ the number of cherry-trees;
$5x=$ the peach-trees;
$6x=$ the apple-trees.

Therefore, $x+4x+5x+6x=144$;
that is, $16x=144$,
or, $x=9$,
$4x=$, etc.

Prob. 45. Four gentlemen entered into a speculation, for which they subscribed 5850 dollars, of which B paid four times as much as A, C paid six times as much as A, and D paid seven times as much as A What did each pay?

Let $x=$ the sum A paid;
$4x=$ the sum B paid;
$6x=$ the sum C paid;
$7x=$ the sum D paid.

Therefore, $x+4x+6x+7x=5850$;

that is, $18x=5850$,

or, $x=325$,

$4x=$, etc.

Prob. 46. The number of days that four workmen were employed were severally as the numbers 1, 2, 3, and 4, and the number of days' work performed by them all was 250. How many days was each workman employed?

Ans. The first was employed 25 days,
the second 50,
the third 75,
the fourth 100.

Prob. 47. The estate of a bankrupt, valued at 33,600 dollars, is to be divided among four creditors according to their respective claims. The debts due to B are double those due to A; those due to C are four times those due to A; and those due to D are five times those due to A. What sum must each receive?

Ans. A receives 2,800 dollars;
B receives 5,600 dollars;
C receives 11,200 dollars;
D receives 14,000 dollars.

Prob. 48. A draper has four pieces of cloth, whose united value is 168 dollars. The value of the second piece is double that of the first; the value of the third is five times that of the first; and the value of the fourth is six times that of the first. What is the value of each?

Ans. The value of the first is 12 dollars;
the second is 24 dollars;
the third is 60 dollars;
the fourth is 72 dollars.

Prob. 49. A grocer has four casks, which together will contain 256 gallons. The capacity of the second is twice that of the first; the capacity of the third is six times that of the first; and the capacity of the fourth is seven times that of the first. What is the capacity of each?

Ans. The capacity of the first is 16 gallons;
the second is 32 gallons;
the third is 96 gallons;
the fourth is 112 gallons.

Prob. 50. Four persons purchased a farm in company for 8840 dollars, of which B paid three times as much as A, C paid four times as much as A, and D paid five times as much as A. What did each pay?

Ans. A paid 680 dollars;
B paid 2040 dollars;
C paid 2720 dollars,
D paid 3400 dollars.

Prob. 51. Four persons, A, B, C, and D, drew prizes in a lottery to the amount of 4875 dollars. B drew three times as much as A, C drew five times as much as A, and D drew six times as much as A. What did each person draw?

Ans. A drew 325 dollars;
B drew 975 dollars;
C drew 1625 dollars;
D drew 1950 dollars.

Prob. 52. An estate of 9775 dollars is to be divided between four heirs, A, B, C, and D. B is to receive three times as much as A, C is to receive six times as much as A, and D is to receive seven times as much as A How much did each receive?

Ans. A received 575 dollars;
B received 1725 dollars;
C received 3450 dollars;
D received 4025 dollars.

Prob. 53. A farmer has 192 fruit-trees in his orchard, consisting of plum-trees, cherry-trees, peach-trees, and apple-trees. The number of cherry-trees is four times the number of plum-trees; the number of peach-trees is five times the number of plum-trees; and the number of apple-trees is six times the number of plum-trees What is the number of each sort?

Ans. There are 12 plum-trees,
48 cherry-trees,
60 peach-trees,
72 apple-trees.

Prob. 54. Four gentlemen entered into a speculation, for which they subscribed 9720 dollars, of which B paid four times as much as A, C paid six times as much as A, and D paid seven times as much as A. What did each pay?

Ans. A paid 540 dollars;
B paid 2160 dollars;
C paid 3240 dollars;
D paid 3780 dollars.

(16.) The pupil can not fail to have remarked a striking similarity between many of the preceding problems. Thus, the first problem requires us to divide the number 6 into two parts, one of which is double the other. Problem 10th requires us to divide the number 36 into two parts, one of which is double the other. It is evident that an infinite number of

QUEST.—In what does Problem 1 differ from Problem 10?

similar problems might be proposed, differing from each other only in the number to be divided. We may discover a general method of solving all these problems by representing the number to be divided by a letter, as a. We shall then have

$$x+2x=a,$$

or, $$3x=a.$$

Hence, $$x=\frac{a}{3}.$$

Thus, in Problem 1, $x=$ one third of 6, which is 2; and in Prob. 10, x equals one third of 36, which is 12.

Again, Prob. 2 requires us to divide the number 60 into two parts, one of which is three times the other Prob. 11 only differs from Prob. 2 in the number proposed to be divided. We can discover a general method of solving all problems of this kind by representing tne number to be divided by a letter, as a. We shall then have

$$x+3x=a,$$

or, $$4x=a.$$

Hence, $$x=\frac{a}{4}.$$

Thus, in Prob. 2, x is one fourth of 60, or 15; and in Prob. 11, x is one fourth of 72, or 18.

Again, Prob. 3 requires us to divide the number 10 into two parts, one of which is four times as great as the other. Prob. 12 requires us to divide the number 15 in a similar manner. We can solve all problems

---

QUEST.—In what does Problem 2 differ from Problem 11? In what does Problem 3 differ from Problem 12?

of this kind in a general manner by representing the proposed number by a. We shall then have

$$x+4x=a,$$

or, $$5x=a.$$

Hence, $$x=\frac{a}{5}.$$

A similar method is applicable to problems 4 and 13, 5 and 14, etc.

There is also an analogy between Problems 1, 2, 3, etc. Prob. 1 requires us to divide a number into two parts, one of which is double the other. Prob. 2 requires us to divide a number into two parts, one of which is three times the other. Prob. 3 requires us to divide a number into two parts, one of which is four times the other, etc. All such problems are included in the following more general problem:

Prob. 55. It is required to divide a number a into two parts, one of which shall be m times as great as the other.

This problem may be solved in the following manner:

Let x represent one of the parts.

Then m times x, which we will write mx, may represent the other part.

And, by the conditions,

$$x+mx=a.$$

We now meet with a difficulty in finding the value of x, because the two terms x and mx can not be united in a single term, as was done in Art. 10. But, since $x+mx$ is equal to x repeated $1+m$ times, we infer that

$$x \text{ must be equal to } \frac{a}{1+m},$$

and this is a general solution of the first eighteen of the preceding problems.

(17.) It will be readily seen that Problems 19 and 28 differ only in the number proposed to be divided; and if we represent the proposed number by a, we shall have a general solution of this class of problems. A similar remark is applicable to Problems 20 and 29; Problems 21 and 30, etc.

There is also an analogy between Problems 19, 20, 21, etc. In each of them it is required to divide a proposed number into three parts, such that the second and third shall be multiples of the first. All these problems are included in the following general problem.

Prob. 56. It is required to divide a number a into three parts, the second of which shall be m times as great as the first, and the third n times as great as the first.

This problem may be solved in the following manner:

Let x represent the first part.

Then mx will represent the second part,
and nx will represent the third part.

And, by the conditions of the problem,

$$x+mx+nx=a.$$

We now encounter the same difficulty as in Prob. 55, because the terms x, mx, and nx can not be united in a single term. Since, however, $x+mx+nx$ is equal to x repeated $1+m+n$ times, we infer that,

$$x \text{ must be equal to } \frac{a}{1+m+n},$$

QUEST.—In what does Problem 19 resemble Problem 28?

and this is a general solution of all the preceding problems from 19 to 36.

(18.) We shall find in a similar manner that Problems 37, 38, 39, etc., are all included in the following general problem.

Prob. 57. It is required to divide a number a into four parts, the second of which shall be m times as great as the first, the third shall be n times as great as the first, and the fourth p times as great as the first.

This problem may be solved in the following manner:

Let x represent the first part.

Then mx will represent the second part,
nx will represent the third part,
and px will represent the fourth part.

And, by the conditions of the problem,

$$x+mx+nx+px=a.$$

And, reasoning in the same manner as in Prob. 56, we conclude that

$$x \text{ must be equal to } \frac{a}{1+m+n+p},$$

and this is a general solution of all the preceding problems from 37 to 54.

(19.) Problems 1 to 54 are called *numerical* problems, and are such problems as occur in common arithmetic. Problems 55 to 57 are *general* problems, and pure Algebra is chiefly confined to problems of this kind, where letters are employed to represent quantities which are supposed to be known, as well as those which are unknown. It becomes necessary, therefore,

QUEST.—What are numerical problems? What are general problems?

to explain the method of performing the operations of addition, subtraction, multiplication, division, etc., upon quantities represented by letters. These methods are essentially the same as practiced in arithmetic, but there is some apparent difference arising from the difficulty we experience in algebra in uniting many terms into one single term, as we do in arithmetic.

Some peculiarities of notation are adopted when numbers are represented by letters.

(20.) The first letters of the alphabet are commonly used to represent *known* quantities, and the last letters those which are *unknown.*

(21.) Quantities preceded by the sign + are called *positive* quantities; those preceded by the sign −, *negative* quantities. When no sign is prefixed to a quantity, + is to be understood. Thus, $a+b-c$, is the same as $+a+b-c$.

(22.) When numbers are represented by letters, multiplication is usually indicated by writing the letters in succession without the interposition of any sign. Sometimes it is indicated by placing a point between the successive letters. Thus, $abcd$ is equivalent to $a\times b\times c\times d$, or $a.b.c.d.$

Thus, if we suppose $a=2$, $b=3$, $c=4$, and $d=5$, we have
$$abcd=2\times3\times4\times5=120.$$

(23.) When two or more quantities are multiplied together, each of them is called a *factor.* Thus, in the expression 7×5, 7 is a factor, and so is 5. In the product abc there are three factors a, b, and c.

QUEST.—How are known quantities represented? What are positive quantities? What are negative quantities? How may multiplication be denoted? What is a factor?

When a quantity is represented by a letter, it is called a *literal* factor, to distinguish it from a *numerical* factor, which is represented by an Arabic numeral. Thus, in the expression $8ab$, 8 is a numerical factor, while a and b are literal factors.

(24.) The symbol > is called the sign of *inequality*, and when placed between two quantities, denotes that one of them is greater than the other, the opening of the sign being turned toward the greater number.

Thus, $4<7$ signifies that 4 is less than 7, and $12>9$ denotes that 12 is greater than 9. So, also, $a>b$ shows that a is greater than b, and $c<d$ shows that c is less than d.

(25.) The *coefficient* of a quantity is the number or letter prefixed to it, showing how often the quantity is to be taken.

Thus, instead of writing $a+a+a+a+a$, which represents 5 a's added together, we write $5a$ where 5 is the coefficient of a. In like manner, $8ab$ signifies eight times the product of a and b. When no coefficient is expressed, 1 is always to be understood. Thus, $1a$ and a signify the same thing.

The coefficient may be a *letter* as well as a figure. In the expression mx, m may be considered as the coefficient of x, because x is to be taken as many times as there are units in m. If m stands for 4, then mx is four times x.

(26.) The products formed by the successive multiplication of the same number by itself are called the *powers* of that number.

QUEST.—What is a literal factor? What is a numerical factor? What is the sign of inequality? What is a coefficient? What are powers?

Thus, $2\times2=4$, which is the second power of 2.
$2\times2\times2=8$, the third power of 2.
$2\times2\times2\times2=16$, the fourth power of 2, etc.
So, also, $3\times3=9$, the second power of 3.
$3\times3\times3=27$, the fourth power of 3, etc.
Also, $a\times a=aa$, the second power of a.
$a\times a\times a=aaa$, the third power, etc.

(27.) For the sake of brevity, powers are usually expressed by writing the root once, with a number above it at the right hand, showing how many times the root is taken as a factor. This number is called the *exponent* of the power.

Thus, instead of
aa, we write a^2, where 2 is the exponent of the power.
aaa, we write a^3, where 3 is the exponent of the power.
$aaaa$, we write a^4, where 4 is the exponent of the power, etc.

When no exponent is expressed, 1 is always understood. Thus, a^1 and a signify the same thing.

Exponents may be attached to figures as well as letters.

Thus, the product of 3 by 3 may be written 3^2, which equals 9.

The product of $3\times3\times3$ may be written 3^3, which equals 27.

The product of $3\times3\times3\times3$ may be written 3^4, which equals 81, etc.

(28.) A *root* of a quantity is a factor which, multiplied by itself a certain number of times, will produce the given quantity.

QUEST.—What is an exponent? What is a root of a quantity?

The symbol $\sqrt{}$ is called the *radical sign*, and, when prefixed to a quantity, denotes that its root is to be extracted. Thus,

$\sqrt[2]{9}$, or simply $\sqrt{9}$, denotes the square root of 9, which is 3.

$\sqrt[3]{64}$ denotes the cube root of 64, which is 4.

$\sqrt[4]{16}$ denotes the fourth root of 16, which is 2.

So, also,

$\sqrt[2]{a}$, or simply $\sqrt{a}$, is the square root of a.

$\sqrt[3]{a}$ denotes the third or cube root of a.

$\sqrt[4]{a}$ denotes the fourth root of a.

$\sqrt[n]{a}$ denotes the nth root of a, when n may represent any number whatever.

(29.) The number placed over the radical sign is called the *index* of the root. Thus 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, and n of the nth root. The index of the square root is usually omitted. Thus, instead of $\sqrt[2]{ab}$, we usually write $\sqrt{ab}$.

(30.) A *vinculum* ———, or a *parenthesis* ( ), indicates that several quantities are to be subjected to the *same operation*.

Thus, $\overline{a+b+c}\times d$, or $(a+b+c)\times d$, denotes that the sum of a, b, and c is to be multiplied by d. But $a+b+c\times d$ denotes that c *only* is to be multiplied by d.

When the parenthesis is used, the sign of multiplication is generally omitted. Thus $(a+b+c)\times d$ is the same as $(a+b+c)d$, or $d(a+b+c)$.

(31.) Every quantity expressed in algebraic an-

QUEST.—What is the radical sign? What is the index of a root? What is a vinculum?

guage, that is, by the aid of algebraic symbols, is called an *algebraic quantity*, or an *algebraic expression* Thus,

$5a$ is the algebraic expression for five times the number a.

$4a^2$ is the algebraic expression for four times the square of the number a.

$6a^2b^3$ is the algebraic expression for six times the square of a, multiplied by the third power of b.

(32.) An algebraic quantity composed of a single term is called a *monomial*.

Thus, $2a$, $3bc$, $5xy^2$, are monomials.

An algebraic expression, consisting of two terms only, is called a *binomial;* one consisting of three terms is called a *trinomial*.

Thus, $3a+5b$ is a binomial, and $a+2bc+5xy$ is a trinomial.

An algebraic expression which is composed of several terms is called a *polynomial*.

Thus, $2a+5b+7c-4d$ is a polynomial.

(33.) The *numerical value* of an algebraic expression is the result obtained when we attribute particular values to the letters.

Suppose the expression is $5ab^2$.

If we make $a=2$ and $b=3$, the value of this expression will be $5\times2\times3\times3=90$.

If we make $a=3$ and $b=4$, the value of the same expression will be $5\times3\times4\times4=240$.

QUEST.—What is an algebraic expression? What is a monomial? What is a binomial? What is a trinomial? What is a polynomial? What is the numerical value of an algebraic expression?

(34.) *Like* or *similar* terms are terms composed of the *same letters* affected with the *same exponents.*

Thus, in the polynomial

$$4ab+9ab+5a^2c-12a^2c,$$

the terms $4ab$ and $9ab$ are similar, and so also are the terms $5a^2c$ and $-12a^2c$.

But in the binomial

$$6ab^2+5a^2b,$$

the terms are *not* similar; for, although they contain the same letters, the same letters are not affected with the same exponents.

(35.) The *reciprocal* of a quantity is the quotient arising from dividing a unit by that quantity.

Thus, the reciprocal of 3 is $\frac{1}{3}$; the reciprocal of a is $\frac{1}{a}$.

The following examples are designed to exercise the pupil upon the preceding definitions and remarks.

(36.) *Examples in which words are to be translated into algebraic symbols.*

Ex. 1. What is the algebraic expression for the following statement? Five times the square of a multiplied by the cube of b?

Ans. $5a^2b^3$.

Ex. 2. Six times the square of a multiplied by the cube of b, diminished by the square of c, multiplied by the fourth power of d.

Ans. $6a^2b^3-c^2d^4$.

Ex. 3. The second power of a increased by twice

QUEST.—What are similar terms? What is the reciprocal of a quantity?

the product of a and b, diminished by c, is equal to nine times d.

Ans. $a^2+2ab-c=9d$.

Ex. 4. Three quarters of x increased by five, is equal to two fifths of b diminished by eleven.

Ans.

Ex. 5. The quotient of three divided by the sum of x and four, is equal to twice b diminished by eight.

Ans.

Ex. 6. One third of the difference between five times x and four, is equal to the quotient of six divided by the sum of a and b.

Ans.

Ex. 7. The quotient arising from dividing the sum of a and b by the product of c and d, is equal to four times the sum of x and y.

Ans.

(37.) *Examples in which the algebraic signs are to be translated into common language.*

Ex. 1. $$\frac{a+b}{c}+mx=\frac{nx}{d}.$$

Ans. **The quotient arising from dividing the sum of a and b by c, increased by the product of m and x, is equal to the quotient arising from dividing n times x by d.**

Ex. 2. $$7a^2+b(c-d)=x+y.$$

Ans.

Ex. 3. $$\frac{a+b}{3+x}+\frac{x}{a}=\frac{c}{d+2}.$$

Ans.

Ex. 4. $$4\sqrt{ab}-17=\frac{b+2c}{d}.$$

Ans.

Ex. 5. $$\frac{x}{2+(b-c)}+\frac{a}{7}=\frac{ab(x+2)}{3}.$$

Ans.

Ex. 6. $$\frac{\sqrt{5b}+x}{2a+1}=3x+14.$$

Ans.

(38.) Find the value of the following algebraic expressions when $a=6$, $b=5$, and $c=4$.

Ex. 1. $$a^2+3ab-c^2.$$

Ans. $36+90-16=110.$

Ex. 2. $$a^2(a+b)-2abc.$$

Ans. 156.

Ex. 3. $$\frac{a^3}{a+3c}+c^2.$$

Ans. 28.

Ex. 4. $$\frac{2ab}{c}+3b+7a.$$

Ans.

Ex. 5. $$\frac{a+2b}{c}+\frac{3a}{b+c}.$$

Ans.

Ex. 6. $$\frac{ab+3c}{2b-3}+6a+\frac{8(a-b)\times(b-c)}{c}.$$

Ans.

Ex. 7. $$8a+\frac{2bc}{\sqrt{2ac+c^2}}.$$

Ans.

Ex. 8. $\sqrt{b^2-ac}+\sqrt{2ac+c^2}.$

Ans.

SECTION II.

ADDITION.

(39.) ADDITION is the connecting of quantities together by means of their proper signs, and incorporating such as can be united into one sum.

If it is required to add a number represented by x to four times itself, we write it

$$x+4x,$$

which may be reduced to $5x$.

If it is required to add a number x to m times itself, we write it $x+mx$,

which two terms can not be united in one.

If it is required to add a number represented by x to three times itself, and also four times itself, we write it $x+3x+4x$,

which may be reduced to $8x$.

If it is required to add a number x to m times itself, and also to n times itself, we write it

$$x+mx+nx,$$

which three terms can not be united in one, and this is called algebraic addition.

(40.) It is convenient to consider this subject under three cases.

CASE I.

When the quantities are similar, and have the same signs.

QUEST.—What is Addition? How many cases are there in Addition? What is case first?

RULE.

Add the coefficients of the several quantities together, and to their sum annex the common letter or letters, prefixing the common sign.

Thus the sum of $3a$ and $5a$ is obviously $8a$.

Ex. 1. What is the sum of $4a$, $6a$, and $9a$?

Ans. $19a$.

Ex. 2. What is the sum of $4xy$, $8xy$, xy, and $3xy$?

Ans. $16xy$.

Ex. 3. What is the sum of $3ab$, $7ab$, ab, and $12ab$?

Ans. $23ab$.

Ex. 4. What is the sum of $2mx$, $9mx$, $4mx$, and $5mx$?

Ans. $20mx$

Add together the following terms:

(5.)	(6.)	(7.)	(8.)
$2b+3x$	$2a+y^2$	$5a+xy$	$3ax+m$
$5b+7x$	$5a+2y^2$	$9a+3xy$	$12ax+7m$
$b+2x$	$9a+3y^2$	$3a+8xy$	$11ax+5m$
$4b+3x$	$4a+6y^2$	$7a+4xy$	$4ax+9m$

Ans. $12b+15x$.

(41.) We proceed in the same manner when all the signs are minus.

Thus the sum of $-3a$ and $-5a$ is $-8a$; for the minus sign before each of the terms shows that they are to be subtracted, not from each other, but from some quantity which is not here expressed; and if $3a$ and $5a$ are to be successively subtracted from the same quantity, it is the same as subtracting at once $8a$.

QUEST.—Give the rule. How do we proceed when all the signs are negative?

Ex. 9. What is the sum of $-3a$, $-7a$, $-11a$, and $-8a$?

Ans. $-29a$.

Ex. 10. What is the sum of $-2bc$, $-12bc$, $-5bc$ and $-bc$?

Ans. $-20bc$.

Ex. 11. What is the sum of $-3ax$, $-6ax$, $-15ax$ and $-4ax$?

Ans. $-28ax$.

Add together the following terms:

(12.)	(13.)	(14.)	(15.)
$3a-2x^2$	$7x-9y$	$3a^2-5ab$	$8ax-12mx$
$5a-7x^2$	$2x-3y$	$6a^2-4ab$	$3ax-2mx$
$6a-12x^2$	$x-8y$	$10a^2-2ab$	$7ax-8mx$
$8a-10x^2$	$12x-4y$	$5a^2-3ab$	$2ax-7mx$
Ans. $22a-31x^2$			

CASE II.

(42.) When the quantities are similar, but have different signs.

RULE.

Add all the positive coefficients together, and also all those that are negative; subtract the least of these results from the greater; to the difference annex the common letter or letters, and prefix the sign of the greater sum.

Thus, instead of $7a-4a$, we may write $3a$, since these two expressions obviously have the same value.

Also, if we have $5a-2a+3a-a$, this signifies that from $5a$ we are to subtract $2a$, add $3a$ to the remainder, and then subtract a from this last sum, the result

QUEST.—What is case second? Give the rule.

of which operation is $5a$. But it is generally most convenient to take the sum of the positive quantities, which in this case is $8a$; then take the sum of the negative quantities, which in this case is $3a$; and we have $8a-3a$, or $5a$, the same result as before.

The pupil must continually bear in mind the remark of Art. 21, that when no sign is prefixed to a quantity, plus is always to be understood.

Ex. 1. What is the sum of $6a$, $-4a$, $3a$, and a?

Ans. $6a$.

Ex. 2. What is the sum of $4xy-2xy+7xy-xy$?

Ans. $8xy$.

Ex. 3. What is the sum of $3abm+2abm-4abm+abm$?

Ans. $2abm$.

Add together the following terms:

	(4.)	(5.)	(6.)	(7.)
	$6x+5ay$	$7ax+4ab$	$-6a^2+2b$	$2ay-7$
	$-3x+2ay$	$3ax-2ab$	$2a^2-3b$	$-ay+8$
	$x-6ay$	$ax+10ab$	$-5a^2-8b$	$2ay-9$
	$2x+ay$	$5ax-6ab$	$4a^2-2b$	$3ay-11$
Ans.	$6x+2ay$			

	(8.)	(9.)	(10.)	(11.)
	$2a^2x-3a$	$8x^2+9ac$	$22h-4y$	$5ab^2+mx$
	a^2x+7a	$7x^2-2ac$	$-16h+2y$	$-7ab^2-3mx$
	$-3a^2x+8a$	$-5x^2+6ac$	$3h+y$	$2ab^2-6mx$
	$7a^2x-a$	x^2-4ac	$9h-3y$	$4ab^2+2mx$
Ans.				

CASE III.

(43.) When some of the quantities are dissimilar.

QUEST.—What is the third case?

RULE.

Collect all the like quantities together, by taking their sums or differences as in the two former cases, and set down those that are unlike, one after the other, with their proper signs.

Unlike quantities can not be united in one term. Thus, $2a$ and $3b$ neither make $5a$ nor $5b$. Their sum can only be written $2a+3b$.

Ex. 1. What is the sum of $3ab+6ax$ and $5ab-2ax+m$?

Ans. $8ab+4ax+m$.

Ex. 2. What is the sum of $4a^2-2ab$ and $5a^2+y+ab$? *Ans.* $9a^2-ab+y$.

Ex. 3. What is the sum of $8ax+2ac$ and $6ax-ac+4ac$? *Ans.* $14ax+5ac$.

Add together the following terms:

(4.)	(5.)	(6.)	(7.)
$2xy-2x^2$	$3xy+2ax$	$3bc-2y$	$5abc+mn$
$3x^2+xy$	$-2xy-ax^2$	$y+b$	$2abc-3mn$
x^2-xy	$-3xy+3ax$	$-2bc+8y$	$2mn-x$
$4x^2-3xy$	$-8xy-ax$	$2y+4bc$	$-4abc+5mn$
Ans. $6x^2-xy$.			

(44.) When several quantities are to be added together, it is most convenient to *write all the similar terms under each other*, as in the following example.

Ex 8 Add together the following terms:

$$11bc+4ad-8x$$
$$8x+7bc-2ad$$
$$2cd-2ad-2bc.$$

QUEST.—Give the rule. What is the most convenient mode of arranging the terms?

These terms may be written thus:

$$\begin{array}{l} 11bc+4ad-8x+2cd \\ 7bc-2ad+8x \\ -2bc-2ad \\ \hline 16bc \qquad\qquad +2cd. \end{array}$$

Sum,

Ex. 9. Add together the following terms:

$$\begin{array}{l} 7m+3n-14p \\ 3a+9n-11m \\ 5p-4m+8n \\ 11n-2b-m \end{array}$$

Ans. $3a-2b-9m+31n-9p.$

Ex. 10. Add together

$$\begin{array}{l} 4a^2b+3c^2d-9m^2n \\ 4m^2n-ab^2+5c^2d \\ 6m^2n-5c^2d+4mn^2 \\ 7mn^2+6c^2d+5m^2n \end{array}$$

Ans. $4a^2b-ab^2+9c^2d+6m^2n+11mn^2.$

Ex. 11. Add together

$$\begin{array}{l} 3b-a-6c \\ 6c-d-3a \\ 3a-2b-3c \\ 5b-8c+9d \\ 17c-6b-7a \end{array}$$

Ans.

Ex. 12. Add together

$$\begin{array}{l} 3am-4xy+8 \\ 10xy+2-am \\ 12am-2xy+6 \end{array}$$

Ans.

Ex 13. Add together

$$\begin{array}{l} 18ax^2-12ab+5m \\ 14ab+6ac+ax^2 \\ 2m-7ax^2+3ab \end{array}$$

Ans.

Ex. 14. Add together

$$\begin{array}{l} 5a+4b-7c \\ 8c+2a-3ax \\ a+2c-8b \\ 7a-4b+3c \end{array}$$

Ans.

Ex. 15. Add together

$$\begin{array}{l} 7ax+4ab+3ab^2 \\ 2ab-3ac+6ax \\ 4ab^2+2ac-25mn \\ -ax+3mn+12ab \end{array}$$

Ans.

Ex. 16 Add together

$$\begin{array}{l} 4xy+m+9ab+16 \\ 3ab+12-2xy+17m \\ 4+xy-12m+2ab \\ 7ab-6m-12+2xy \\ 25+3m-6xy+5ab \end{array}$$

Ans.

Ex 17. Add together

$$\begin{array}{l} 7abc+3xy+2mn \\ 14+6abc+17xy \\ 6mn+10xy-3abc \\ -3xy+15-2abc \end{array}$$

Ans.

Ex. 18. Add together

$$\begin{array}{l} 2ab^2 + 3ac^2 + 9b^2x \\ -3x^2 + 2ab^2 - b^2x \\ -22ac^2 - 10x^2 - 4ab^2 \\ 19ac^2 - 8b^2x + 9x^2 \end{array}$$

Ans.

Ex. 19. Add together

$$\begin{array}{l} 4a^2b - 7ab + 2x \\ -5a^2b + 3x - 2ab \\ 9x - 4ab + 2ab \end{array}$$

Ans.

Ex. 20. Add together

$$\begin{array}{l} 12mx + 3an - 25 \\ -2an + 4mx + y \\ +16 + 4an - 5mx \end{array}$$

Ans.

(45.) It must be observed that the term addition is used in a more extended sense in algebra than in arithmetic. In arithmetic, where all quantities are regarded as positive, addition implies *augmentation.* The sum of two quantities will therefore be numerically greater than either quantity. Thus the sum of 7 and 5 is 12, which is numerically greater than either 5 or 7.

But in algebra we consider negative as well as positive quantities; and by the sum of two quantities we mean their aggregate, regard being paid to their signs. Thus the sum of +7 and −5 is +2, which is numerically less than either 7 or 5. So, also, the sum of

---

QUEST.—What is the difference between arithmetical and algebraic addition?

$+a$ and $-b$ is $a-b$. In this case the algebraic sum is numerically the *difference* of the two quantities.

This is one instance among many in which the same terms are used in a much more general sense in the higher mathematics than they are in arithmetic.

SECTION III.

SUBTRACTION.

(46.) SUBTRACTION is the taking of one quantity from another; or it is finding the difference between two quantities or sets of quantities.

Thus, if it is required to subtract 17 from 25, we may write it $25-17$, which equals 8.

So, also, if it is required to subtract $5a$ from $8a$, we nay write it $8a-5a$, which equals $3a$.

If it is required to subtract $5b$ from $8a$, we write it

$$8a-5b,$$

and these, being unlike terms, can not be united in one single term. Hence, *if the quantities are positive and similar, subtract the coefficient of the subtrahend from the coefficient of the minuend, and to their difference annex the literal part. If the quantities are not similar, the subtraction can only be indicated by the usual sign.*

	(1.)	(2.)	(3.)	(4.)
From	$25a$	$16ab^2$	$12abx$	$29ax$
Subtract	$12a$	$3ab^2$	$5abx$	$17ax$
Remainder,	$13a$	$13ab^2$	$7abx$	$12ax$

QUEST.—What is subtraction? If the quantities are positive and similar, how do we proceed? When the quantities are not similar?

	(5.)	(6.)	(7.)	(8.)
From	$13mx^2$	$25ab^2x$	$21a^2b^2c$	$32axy$
Take	$5mx^2$	$19ab^2x$	$8a^2b^2c$	$13axy$
Rem.	$8mx^2$			

	(9.)	(10.)	(11.)	(12.)
From	$7mx$	$8ab^2$	$6bx$	$12xy$
Take	$5ab$	$7b$	$24c$	16
Rem.	$7mx-5ab$			

(47.) Let us now consider the case in which the quantities are not all positive; and let it be required to subtract $8-3$ from 15.

We know that $8-3$ is equal to 5, and 5 subtracted from 15 leaves 10.

The result, then, must be 10. But, to perform the operation on the numbers as they were given, we first subtract 8 from 15 and obtain 7. This result is *too small* by 3, because the number 8 is *larger* by 3 than the number which was required to be subtracted. Therefore, in order to correct this result, the 3 must be added, and we have

$$15-8+3=10, \text{ as before.}$$

Again, let it be required to subtract $c-d$ from $a-b$. It is plain that if the part c were alone to be subtracted, the remainder would be

$$a-b-c.$$

But as the quantity actually proposed to be subtracted is *less* than c by the units in d, *too much* has been taken away by d, and therefore the true remainder will be greater than $a-b-c$ by the units in d, and will hence be expressed by

---

QUEST.—When the quantities are not all positive?

$$a-b-c+d,$$

where the signs of the last two terms are both *contrary* to what they were given in the subtrahend.

(48.) Hence we deduce the following general

Rule.

Conceive the signs of all the terms of the subtrahend to be changed from + to − or from − to +, and then collect the terms together as in the several cases of addition.

It is better, in practice, to leave the signs of the subtrahend *unchanged*, and simply *conceive* them to be changed; that is, treat the quantities as if the signs were changed;* for otherwise, when we come to revise the work, to detect any error in the operation, we might often be in doubt as to what were the signs of the quantities as originally proposed.

Examples.

Ex. 1. From $7x^2+4y$ take $3x^2-2y$.

Ans. $4x^2+6y$.

	(2.)	(3.)	(4.)
From	$5a^2-2b$	$5xy+8x-2$	$x^2+2xy+y^2$
Take	$2a^2+5b$	$3xy-8x-7$	$x^2-2xy+y^2$
Rem.	$3a^2-7b$		

	(5.)	(6.)	(7.)
From	$3a^2+\ ax+2x^2$	$10-8x-3xy$	$4ax-2x^2y$
Take	$2a^2-4ax+\ x^2$	$3-\ x-\ xy$	$2ax-5xy^2$
Rem.	$a^2+5ax+\ x^2$		

Quest.—Give the general rule for subtraction. Is it best actually to change the signs of the subtrahend?

	(8.)	(9.)	(10.)
From	$5a+4b-2c$	$7d+11xy$	$2y^2-16x^2$
Take	$3a+2b+c$	$5d-4xy$	$6y^2-18x^2$
Rem.	$2a+2b-3c$		

	(11.)	(12.)
From	$6aby-4xy+4xz$	$14a^2x+19ax^2+5a^2x^2$
Take	$-3aby+5xz+3xy$	$15a^2x+11ax^2-15a^2x^2$
Rem.	$9aby-xz-7xy$	

(49.) Subtraction may be *proved*, as in arithmetic, by adding the remainder to the subtrahend. The sum should be equal to the minuend.

The term subtraction, it will be perceived, is used in a more general sense in algebra than in arithmetic. In arithmetic, where all quantities are regarded as positive, a number is always *diminished* by subtraction. But in algebra, the difference between two quantities may be numerically greater than either. Thus the difference between $+a$ and $-a$ is $2a$.

(50.) The distinction between positive and negative quantities may be illustrated by the scale of a thermometer. The degrees above zero are considered positive, and those below zero negative. From five degrees above zero to five degrees below zero, the numbers stand thus:

$$+5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5.$$

The difference between five degrees above zero and five degrees below zero is ten degrees, which is numerically the *sum* of the two quantities.

QUEST.—How may subtraction be proved? What is the difference between arithmetical and algebraic subtraction? Illustrate the distinction between positive and negative quantities.

(51.) In many cases the terms positive and negative are merely *relative*. They indicate some sort of *opposition* between two classes of quantities, such that if one class should be *added*, the other ought to be *subtracted*. Thus, if a ship sail alternately northward and southward, and the motion in one direction is called *positive*, the motion in the opposite direction should be considered *negative*.

Suppose a ship, setting out from the equator, sails northward 50 miles, then southward 27 miles, then northward 15 miles, then southward again 22 miles, and we wish to know the last position of the ship. If we call the northerly motion +, the whole may be expressed algebraically thus:

$$+50-27+15-22,$$

which reduces to +16. The positive sign of the result indicates that the ship was 16 miles *north* of the equator.

Suppose the same ship sails again 8 miles north, then 35 miles south, the whole may be expressed thus:

$$+50-27+15-22+8-35,$$

which reduces to −11. The negative sign of the result indicates that the ship was now 11 miles *south* of the equator.

In this example we have considered the northerly motion + and the southerly motion −; but we might, without impropriety, have considered the southerly motion + and the northerly motion −. It is, however, indispensable that we adhere to the *same system* throughout, and retain the proper sign of the result,

---

QUEST.—Sometimes the terms positive and negative are merely relative. Illustrate this by the example of a ship as this sign shows whether the ship was at any time north or south of the equator.

In the same manner, if we consider easterly motion +, westerly motion must be regarded as −, and vice versa. And, generally, when quantities which are estimated in different directions enter into the same algebraic expression, those which are measured in one direction being treated as +, those which are measured in the opposite direction must be regarded as −.

So, also, in estimating a man's property, *gains* and *losses*, being of an opposite character, must be affected with different signs. Suppose a man with a property of 1000 dollars loses 300 dollars, afterward gains 100, and then loses again 400 dollars, the whole may be expressed algebraically thus:

$$+1000-300+100-400,$$

which reduces to +400. The + sign of the result indicates that he has now 400 dollars remaining in his possession. Suppose he further gains 50 dollars and then loses 700 dollars. The whole may now be expressed thus:

$$+1000-300+100-400+50-700,$$

which reduces to −250. The − sign of the result indicates that his losses exceed the sum of all his gains and the property originally in his possession; in other words, he owes 250 dollars more than he can pay, or, in common language, he is 250 dollars *worse than nothing*.

(52.) It is sometimes sufficient merely to *indicate* the subtraction of a polynomial without actually per-

QUEST.—Illustrate the same principle by a case of gain and loss. How do we indicate the subtraction of a polynomial?

forming the operation. This is done by inclosing the polynomial in a parenthesis, and prefixing the sign $-$. Thus,

$$5a-3b-(3a-2b),$$

signifies that the entire quantity $3a-2b$ is to be subtracted from $5a-3b$. The subtraction is here merely *indicated.* If we actually perform the operation, the expression becomes

$$5a-3b-3a+2b,$$

or,

$$2a-b.$$

According to this principle, polynomials may be written in a variety of forms. Thus,

$$a-b-c+d,$$

is equivalent to $a-(b+c-d)$,

or to $a-b-(c-d)$,

or to $a+d-(b+c)$.

Examples for Practice.

Ex. 1. From $10ax+y$ take $3ax-y$.

Ans. $7ax+2y$.

Ex. 2. From $17mx^2+12$ take $17mx^2+12-b$.

Ans. $+b$.

Ex. 3. From $12ab^2x^2-y^2$ take $3ab^2x^2+y^2$.

Ans. $9ab^2x^2-2y^2$.

Ex. 4. From $a+b$ take $a-b$.

Ans. $2b$.

Ex. 5. From $5a^2+b^2+2c^2-15$ take $12+b^2+5a^2$.

Ans.

Ex. 6. From $17x+4a-3b+25$ take $12+2b-3a+4x$.

Ans.

Quest.—Give some of the different forms in which a polynomial may be written.

Ex. 7. From $3a+2b-7c+14y$ take $3y+4c-b+a$.

Ans.

Ex. 8. From $28ax^2-16a^2x^2+25a^2x-13a^4$ take $18ax^2+20a^2x^2-24a^2x-7a^4$.

Ans.

Ex. 9. From $8a^2xy-5bx^2y+17cxy^2-9y^5$ take $a^2xy+3bx^2y-13cxy^2+20y^5$.

Ans.

Ex. 10. From $10ax^2+13by^2-17bc$ take $4bc-3by^2+6ax^2$.

Ans.

Ex. 11. From $25x+32xy-6a$ take $2a+17-19xy+3x$.

Ans.

Ex. 12. From $12x^2$ take $24x+3x-7x+16x$.

Ans.

Ex. 13. From $24a+17b$ take $20+3b-2a+7a-6b$.

Ans.

Ex. 14. From $34-6ab+2y-5x$ take $2x+3ab-16+y$.

Ans.

Ex. 15. From $a^2+abc-6$ take $6+abc-a^2$.

Ans.

SECTION IV.

MULTIPLICATION.

(53.) **Multiplication** is repeating the multiplicand as many times as there are units in the multiplier.

Case I.

When both the factors are monomials.

If the quantity a is to be repeated five times, we may write it thus:

$$a+a+a+a+a,$$

which is equal to $5a$; that is, a multiplied by 5 is equal to $5a$.

If b is to be repeated six times, we may write it

$$b+b+b+b+b+b,$$

which is equal to $6b$.

If x is to be repeated any number of times, for instance as many times as there are units in a, we may write it ax, which signifies a times x, or x multiplied by a.

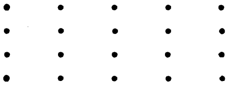
Again, if ab is to be repeated four times, we may write it $$ab+ab+ab+ab,$$ which is equal to $4ab$, or four times the product ab.

(54.) When several quantities are to be multiplied together, the result will be the same in whatever *order* the multiplication is performed. Thus, let five dots

---

Quest.—What is multiplication? What is case first? Is it material in what order the multiplication be performed?

be arranged upon a horizontal line, and let there be formed four such parallel lines.



Then it is plain that the number of units in the table is equal to the five units of the horizontal line repeated as many times as there are units in a vertical column; that is, to the product of 5 by 4. But this sum is also equal to the four units of a vertical line repeated as many times as there are units in a horizontal line; that is, to the product of 4 by 5. Therefore the product of 5 by 4 is equal to the product of 4 by 5. For the same reason, $2\times3\times4$ is equal to $2\times4\times3$, or $4\times3\times2$, or $3\times4\times2$, the product in each case being 24. So, also, if a, b, and c represent any three numbers, we shall have abc equal to bca or cab. It is, however, generally most convenient to arrange the letters in *alphabetical* order.

If a man earn $4x$ dollars a month, how much will he earn in $5y$ months?

Here we must repeat $4x$ dollars as many times as there are units in $5y$; hence the product is

$$4x\times5y.$$

which is equal to $20xy$.

(55.) Hence, for the multiplication of monomials we have the following

RULE.

Multiply the coefficients of the two terms together, and to the product annex all the different letters in succession.

QUEST.—Give the rule for the multiplication of monomials.

Examples.

	(1.)	(2.)	(3.)	(4.)
Multiply	$24a$	$7ab$	$7axy$	$3ay$
by	$3b$	$8cd$	$6ay$	$8bm$
Product	$72ab$	$56abcd$		

(56.) We have seen, in Art. 27, that when the same letter appears several times as a factor in a product, this is briefly expressed by means of an exponent. Thus aaa is written a^3, the number three showing that a enters three times as a factor. Hence, if the same letters are found in two monomials which are to be multiplied together, the expression for the product may be abbreviated by adding the exponents of the same letters. Thus, if we are to multiply a^2 by a^3, we find a^3 equivalent to aaa, and a^2 to aa. Therefore the product will be $aaaaa$, which may be written a^5, a result which we might have obtained at once by adding together 2 and 3, the exponents of the common letter a.

Hence, since every factor of both multiplier and multiplicand must appear in the product, we have the following

RULE FOR THE EXPONENTS.

Powers of the same quantity may be multiplied by adding their exponents.

Examples.

	(1.)	(2.)	(3.)
Multiply	$2a^3b^2c$	$2a^2b^3c^4$	$8a^2bc^2$
by	$8abc^3$	$5a^5bc^3$	$7abcd^2$
Product	$16a^4b^3c^4$		

QUEST.—Give the rule for the exponents in multiplication.

	(4.)	(5.)	(6.)
Multiply	$5a^4b^3c^3$	$9a^3b^3c^3$	$25a^2b^3cd$
by	$7a^3b^4c^3d$	$8a^2b^2d$	$9abc^2$
Product	$35a^7b^7c^6d$		

	(7.)	(8.)	(9.)
Multiply	$6a^2b^2x$	$11a^2b^3xy$	$17ab^2c^3d^4$
by	$7bx^5$	$11a^2b^3xy$	$9a^5b^4c^3d^2$
Product	$42a^2b^3x^6$		

	(10.)	(11.)	(12.)
Multiply	$16abxy$	$27a^5b^4c^3d^2$	$9a^3b^4x^5$
by	$7a^3b^3c^3xy$	$15a^2xy^2$	$8a^2bxy^3$
Product	$112a^4b^4c^3x^2y^2$		

	(13.)	(14.)	(15.)
Multiply	$15a^2b^3y$	$17amxy$	$54a^7b^5c^3x$
by	$7abx$	$9abx$	$9ab^3c^4x^2y^3$
Product	$105a^3b^4xy^4$		

Case II.

(57.) *When the multiplicand is a polynomial.*

If $a+b$ is to be multiplied by c, this implies that the sum of the units in a and b is to be repeated c times; that is, the units in b repeated c times must be added to the units in a repeated also c times. Hence we deduce the following

Rule.

Multiply each term of the multiplicand separately by the multiplier, and add together the products.

Quest.—What is the second case in multiplication? Give the rule.

Examples.

	(1.)	(2.)	(3.)
Multiply	$3a+2b$	a^2+2x+1	$3y^3+5xy^2+2$
by	$4a$	$4x$	xy
Product	$12a^2+8ab$		

	(4.)	(5.)	(6.)
Multiply	$3x^3+xy+2y^2$	$16d+7xy^2$	$7bm+4x^2y^3$
by	$5x^2y$	$5ab$	$4a^2bc^3$
Product	$15x^5y+5x^3y^2+10x^2y^3$		

CASE III.

(58.) ***When both the factors are polynomials.***

If $a+b$ is to be multiplied by $c+d$, this implies that the quantity $a+b$ is to be repeated as many times as there are units in the sum of c and d; that is, we are to multiply $a+b$ by c and d successively, and add the partial products. Hence we deduce the following

RULE.

Multiply each term of the multiplicand by each term of the multiplier separately, and add together the products.

Examples.

	(1.)	(2.)	(3.)
Multiply	$a+b$	$3x+2y$	$ax+b$
by	$a+b$	$2x+3y$	$cx+d$
Product	$a^2+2ab+b^2$		

(59.) When several terms in the product are *similar*, it is most convenient to set them under each other, and then unite them by the rules for addition.

QUEST.—What is the third case in multiplication? Give the rule. When several terms are similar, how do we proceed?

	(4.)	(5.)	(6.)
Multiply	$3a+x$	a^2+2b	x^2+2x+4
by	$2a+4x$	$a+b^2$	$x+2$
Product	$6a^2+14ax+4x^2$		

	(7.)	(8.)	(9.)
Multiply	$2x^2+5y$	x^2+xy+y^2	$ab+cd$
by	$2x^2+5y$	x^2+xy+y^2	$mx+ny$
Product	$4x^4+20x^2y+25y^2$		

(60.) The examples hitherto given in multiplication have been confined to *positive* quantities, and the products have all been positive. We must now establish a general rule for the signs of the product.

First. If $+a$ is to be multiplied by $+b$, this signifies that $+a$ is to be repeated as many times as there are units in b, and the result is $+ab$. That is, a plus quantity multiplied by a plus quantity gives a plus result.

Secondly. If $-a$ is to be multiplied by $+b$, this signifies that $-a$ is to be repeated as many times as there are units in b. Now $-a$, taken twice, is obviously $-2a$, taken three times is $-3a$, etc.; hence, if $-a$ is repeated b times, it will make $-ba$, or $-ab$. That is, a minus quantity multiplied by a plus quantity gives minus.

Thirdly. To determine the sign of the product when the multiplier is a minus quantity, let it be proposed to multiply 8−5 by 6−2. By this we understand that the quantity 8−5 is to be repeated as many times as there are units in 6−2. If we multiply 8−5 by 6,

---

QUEST.—What is the product of $+a$ by $+b$? What is the product of $-a$ by $+b$? When the multiplier is a minus quantity?

we obtain $48-30$; that is, we have repeated $8-5$ six times. But it was only required to repeat the multiplicand four times, or $(6-2)$. We must therefore *diminish* this product by twice $(8-5)$, which is $16-10$; and this subtraction is performed by changing the signs of the subtrahend; hence we have

$$48-30-16+10,$$

which is equal to 12. This result is obviously correct, for $8-5$ is equal to 3, and $6-2$ is equal to 4; that is, it was required to multiply 3 by 4, the result of which is 12, as found above.

(61.) In order to generalize this reasoning, let it be proposed to multiply $a-b$ by $c-d$.

If we multiply $a-b$ by c, we obtain $ac-bc$. But it was proposed to take $a-b$ only as many times as there are units in the *difference* between c and d; therefore the product $ac-bc$ is too large by $a-b$ taken d times; that is, to have the true product, we must subtract d times $a-b$ from $ac-bc$. But d times $a-b$ is equal to $ad-bd$, which subtracted from $ac-bc$, gives

$$ac-bc-ad+bd.$$

Thus we see that $+a$ multiplied by $-d$ gives $-ad$, and $-b$ multiplied by $-d$ gives $+bd$. Hence a plus quantity multiplied by a minus quantity gives minus; and a minus quantity multiplied by a minus quantity gives plus.

(62.) The preceding results may be briefly expressed as follows:

+ multiplied by +, and − multiplied by −, give +.

+ multiplied by −, and − multiplied by +, give −.

QUEST.—What is the product of $a-b$ by $c-d$? How may these results be expressed?

Or, the product of two quantities having the *same sign*, has the sign *plus*; the product of two quantities having *different signs*, has the sign *minus*.

(63.) Hence all the cases of multiplication are comprehended in the following

Rule.

Multiply each term of the multiplicand by each term of the multiplier, and add together all the partial products, observing that like signs require + in the product, and unlike signs −.

Examples.

	(1.)	(2.)	(3.)
Multiply	$3a^3-2b^3$	x^2-2x+4	$4a+2b$
by	$a-b$	$x-2$	$4a-2b$
Product	$3a^4-2ab^3-3a^3b+2b^4$		

	(4.)	(5.)	(6.)
Multiply	a^5+a^3+a	$a^2-2ab+b^2$	a^2+ab+b^2
by	a^2-1	$2a-3b$	a^2-ab+b^2
Product	a^7-a		

Ex. 7. Multiply $6a^2+12ax-6x^2$ by $2a^2-4ax-2x^2$.

Ans. $12a^4-72a^2x^2+12x^4$.

Ex. 8. Multiply $x^2-2xy-3$ by $5x^2+10xy+15$.

Ans. $5x^4-20x^2y^2-60xy-45$.

Ex. 9. Multiply $2x^3+2x^2y^2+2y^3$ by $3x^3-3x^2y^2-3y^3$.

Ans. $6x^6-6x^4y^4-12x^3y^5-6y^6$.

Ex. 10. Multiply $a^3-2b^3+c^3$ by a^3-b^3.

Ans.

Ex. 11. Multiply $5a^4-2a^3b+4a^2b^2$ by $a^3-4a^2b+2b^3$.

Ans. $5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.

QUEST.—Give the general rule for multiplication.

Ex. 12. Multiply $4a^3-5a^2b-8ab^2+2b^3$ by $2a^2-3ab-4b^2$.

Ans. $8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5$.

Ex. 13. Multiply $3a^2-5bd+ef$ by $-5a^2+4bd-8ef$.

Ans. $-15a^4+37a^2bd-29a^2ef-20b^2d^2+44bdef-8e^2f^2$.

Ex. 14. Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.

Ans. x^6-2x^3+1.

Ex. 15. Multiply $14a^3c-6a^2bc+c^2$ by $14a^3c+6a^2bc-c^2$.

Ans.

Ex. 16. Multiply $3a^3+35a^2b-17ab^2-13b^3$ by $3a^2+26ab-57b^2$.

Ans.

(64.) For many purposes it is sufficient merely to *indicate* the multiplication of two polynomials, without actually performing the operation. This is effected by inclosing the quantities in parentheses, and writing them in succession, with or without the interposition of any sign.

Thus $(a+b+c)\,(d+e+f)$ signifies that the sum of a, b, and c is to be multiplied by the sum of d, e, and f.

When the multiplication is actually performed, the expression is said to be *expanded.*

(65.) The following theorems are of such extensive application that they should be carefully committed to memory.

THEOREM I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

QUEST.—How may we indicate the multiplication of polynomials? What is the square of the sum of two quantities?

Thus, if we multiply $a+b$
by $a+b$

$$\begin{array}{r} a^2+ab \\ ab+b^2 \\ \hline \end{array}$$

we obtain the product $a^2+2ab+b^2$.
Hence, if we wish to obtain the square of a binomial, we can write out the terms of the result at once, according to this theorem, without the necessity of performing an actual multiplication.

Examples.

Ex. 1. $(2a+b)^2=4a^2+4ab+b^2$.
Ex. 2. $(3a+3b)^2=9a^2+18ab+9b^2$.
Ex. 3. $(4a+3b)^2=16a^2+24ab+9b^2$.
Ex. 4. $(5a^2+b)^2=25a^4+10a^2b+b^2$.
Ex. 5. $(5a^2+7ab)^2=25a^4+70a^3b+49a^2b^2$.
Ex. 6. $(5a^3+8a^2b)^2=25a^6+80a^5b+64a^4b^2$.

This theorem deserves particular attention, for one of the most common mistakes of beginners is to call the square of $a+b$ equal to a^2+b^2.

THEOREM II.

(66.) *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

Thus, if we multiply $a-b$
by $a-b$

$$\begin{array}{r} a^2-ab \\ -ab+b^2 \\ \hline \end{array}$$

we obtain the product $a^2-2ab+b^2$.

QUEST.—Illustrate by an example. What mistake do beginners frequently commit? What is the square of the difference of two quantities?

Examples.

Ex. 1. $(2a-3b)^2=4a^2-12ab+9b^2$.

Ex. 2. $(5ab-2x)^2=25a^2b^2-20abx+4x^2$.

Ex. 3. $(8a^2-3x)^2=64a^4-48a^2x+9x^2$.

Ex. 4. $(6a^2-2b)^2=36a^4-24a^2b+4b^2$.

Ex. 5. $(7a^2-10ab)^2=49a^4-140a^3b+100a^2b^2$.

Ex. 6. $(7a^2b^2-12ab)^2=49a^4b^4-168a^3b^3+144a^2b^2$.

Here, also, beginners often commit the mistake of putting the square of $a-b$ equal to a^2-b^2.

THEOREM III.

(67.) *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus, if we multiply $a+b$

by $a-b$

$$\begin{array}{l} a^2+ab \\ \quad\; -ab-b^2 \\ \hline a^2 \quad\;\; -b^2 \end{array}$$

we obtain the product a^2-b^2.

Examples.

Ex. 1. $(3a+4b)(3a-4b)=9a^2-16b^2$.

Ex. 2. $(6ab+3x)(6ab-3x)=36a^2b^2-9x^2$.

Ex. 3. $(7a+2b)(7a-2b)=49a^2-4b^2$.

Ex. 4. $(8a+7bc)(8a-7bc)=64a^2-49b^2c^2$.

Ex. 5. $(5a^2+6b^3)(5a^2-6b^3)=25a^4-36b^6$.

Ex. 6. $(5x^2y+3xy^2)(5x^2y-3xy^2)=25x^4y^2-9x^2y^4$.

The student should be drilled upon examples like the preceding until he can produce the results mentally with as great facility as he could read them if exhibited upon paper.

QUEST.—Illustrate by examples. What is the product of the sum and difference of two quantities? Illustrate by examples.

The utility of these theorems will be the more apparent the more complicated the expressions to which they are applied. Frequent examples of their application will be seen hereafter.

(68.) The same theorems will enable us to resolve many complicated expressions into their factors.

Ex. 1. Resolve $a^2+2ab+b^2$ into its factors.

Ans. $(a+b)(a+b)$.

Ex. 2. Resolve n^2+2n+1 into its factors.

Ans. $(n+1)(n+1)$.

Ex. 3. Resolve $a^2-2ab+b^2$ into its factors.

Ans. $(a-b)(a-b)$.

Ex. 4. Resolve $a^2-6ab+9b^2$ into its factors.

Ans. $(a-3b)(a-3b)$.

Ex. 5. Resolve a^2-b^2 into its factors.

Ans. $(a+b)(a-b)$.

Ex. 6. Resolve a^4-b^4 into its factors.

Ans. $(a^2+b^2)(a^2-b^2)$.

QUEST.—How may complicated expressions be resolved into factors?

SECTION V.

DIVISION.

(69.) *Division consists in finding how many times one quantity is contained in another.* The quantity to be divided is called the *dividend;* and the *quotient* shows how many times the divisor is contained in the dividend.

When we have obtained the quotient, we may verify the result by multiplying the divisor by the quotient—the product should be equal to the dividend. Hence we may regard the dividend as the product of two factors, viz., the divisor and quotient—of which one is given, that is, the divisor; and it is required to find the other factor, which we call the quotient.

CASE I.

(70.) *When the divisor and dividend are both monomials.*

Suppose we have 72 to be divided by 8. We must find such a factor as multiplied by 8 will give exactly 72. We perceive that 9 is such a number, and therefore 9 is the quotient obtained when we divide 72 by 8.

Also, if we have ab to be divided by a, it is evident that the quotient will be b; for a multiplied by b gives the dividend ab.

QUEST.—What is the object of division? What is the dividend? What is the divisor? What is the quotient? What is case first?

If we divide $60x$ by 5, we obtain $12x$, for $12x$ multiplied by 5 gives $60x$.

So, also, $12mn$ divided by $3m$ gives $4n$.

Again, suppose we have a^5 to be divided by a^2. We must find a number which, multiplied by a^2, will produce a^5. We perceive that a^3 is such a number; for, according to Art. 56, we multiply a^3 by a^2 by adding the exponents, 2 and 3 making 5. That is, the exponent 3 of the quotient is found by subtracting 2, the exponent of the divisor, from 5, the exponent of the dividend. Hence we derive the following

RULE OF EXPONENTS IN DIVISION.

(71.) *A power is divided by another power of the same root, by subtracting the exponent of the divisor from that of the dividend.*

Examples.

	(1.)	(2.)	(3.)	(4.)
Divide	a^8	a^7	b^6	c^9
by	a^3	a^2	b^3	c^4
Quotient	a^5			

	(5.)	(6.)	(7.)	(8.)
Divide	h^7	x^8	y^{10}	d^8
by	h^4	x^5	y^2	d
Quotient	h^3			

(72.) Let it be required to divide $48a^5$ by $6a^2$. We must find a quantity which, multiplied by $6a^2$, will produce $48a^5$. Such a quantity is $8a^3$; for, according to Arts. 55 and 56, $8a^3 \times 6a^2$ is equal to $48a^5$. There-

QUEST.—How are powers divided?

fore, $48a^5$ divided by $6a^2$ gives for a quotient $8a^3$; that is, we have divided 48, the coefficient of the dividend, by 6, the coefficient of the divisor, and have subtracted the exponent of the divisor from the exponent of the dividend.

Hence, for the division of monomials, we have the following

Rule.

1. *Divide the coefficient of the dividend by the coefficient of the divisor.*

2. *Subtract the exponent of each letter in the divisor from the exponent of the same letter in the dividend.*

Examples.

1. Divide $20x^3$ by $4x$. *Ans.* $5x^2$.
2. Divide $25a^3x$ by $5a^2$. *Ans.* $5ax$.
3. Divide $16b^2x^2$ by $4bx$. *Ans.* $4bx$.
4. Divide $72a^5x^2y^4$ by $6a^3y^2$. *Ans.* $12a^2x^2y^2$.
5. Divide $15x^4y^2$ by $3x^2y$. *Ans.* $5x^2y$.
6. Divide $77a^7bx^2$ by $7a^5x$. *Ans.* $11a^2bx$.
7. Divide $84b^6x^3y^5$ by $7b^3xy^4$ *Ans.* $12b^3x^2y$.
8. Divide $48a^8bc$ by $6a^7b$. *Ans.* $8ac$.
9. Divide $36x^8y^4z^6$ by $4xyz$. *Ans.*
10. Divide $42a^6bcd$ by $7ab$. *Ans.*
11. Divide $88b^6cy^2$ by $11by$. *Ans.*
12. Divide $64a^4bdx^2$ by $8abx$. *Ans.*
13. Divide $99x^4y^3z^7$ by $9xyz$. *Ans.*
14. Divide $27a^2by^8$ by $3ay$. *Ans.*
15. Divide $96b^2x^5y^6$ by $8bxy$. *Ans.*
16. Divide $54abx^7$ by $6ax$. *Ans.*

QUEST.—Give the rule for the division of monomials.

When the division can not be exactly performed, the quotient may be expressed in the form of a fraction, and this fraction may be reduced to its lowest terms according to a method to be explained in Art. 79.

Signs in Division.

(73.) The proper *sign* to be prefixed to a quotient may be deduced from the principles already established for multiplication, since the product of the divisor and quotient must be equal to the dividend. Hence,

$$\text{because}\ \left.\begin{array}{r} +a\times+b=+ab \\ -a\times+b=-ab \\ +a\times-b=-ab \\ -a\times-b=+ab \end{array}\right\}\ \text{therefore}\ \left\{\begin{array}{l} +ab\div+b=+a \\ -ab\div+b=-a \\ -ab\div-b=+a \\ +ab\div-b=-a \end{array}\right.$$

Hence we have the following

Rule for the Signs.

When both the dividend and divisor have the same sign, the quotient will have the sign +; when they have different signs, the quotient will have the sign −.

Examples.

1. Divide $-15y^2$ by $3y$.

Here it is plain that the answer must be $-5y$; for $3y\times-5y=-15y^2$.

2. Divide $40a^7bd$ by $-5a^3b$. *Ans.* $-8a^4d$.

3. Divide $-58x^8y^5$ by $2x^4y^2$. *Ans.* $-29x^4y^3$.

4. Divide $-18a^5c^4d^7x^5$ by $-3a^2c^3x^4$.
Ans. $+6a^3cd^7x$.

Quest.—When the division can not be exactly performed, what is to be done? How do we determine the proper sign to be prefixed to the quotient? Give the rule for the signs.

5. Divide $77a^2b^3x^4y^3$ by $-7a^2b^2y^2$.

Ans. $-11bx^4y$.

6. Divide $96b^4c^6d^2$ by $-8b^3c^5$. *Ans.*

7. Divide $-64a^6b^6c^7d$ by $8a^3bd^3$. *Ans.*

8. Divide $88x^6y^5z$ by $-11x^5y^2$. *Ans.*

9. Divide $-72a^4c^2d^5x^3$ by $-6a^2cx^2$. *Ans.*

10. Divide $84b^7c^3y^8$ by $-12b^3c^2y^5$. *Ans.*

Case II.

(74.) *When the divisor is a monomial and the dividend a polynomial.*

We have seen, Art. 57, that when a single term is multiplied into a polynomial, the former enters into *every term* of the latter.

Thus, if we multiply a by $a+b$, we obtain for a product a^2+ab.

Hence, if we divide a^2+ab by a, the quotient must be $a+b$.

Therefore, when the divisor is a monomial and the dividend a polynomial, we have the following

Rule.

Divide each term of the dividend by the divisor, as in the former Case.

Examples.

1. Divide $3x^3+6x^2+3ax-15x$ by $3x$.

Ans. $x^2+2x+a-5$.

2. Divide $3abc+12abx-9a^2b$ by $3ab$.

Ans. $c+4x-3a$.

QUEST.—What is the second case? Give the rule when the dividend is a polynomial.

3. Divide $40a^3b^3+60a^2b^2-17ab$ by $-ab$.

Ans. $-40a^2b^2-60ab+17$.

4. Divide $15a^2bc-10a^2cx^2+5a^3c^2d^2$ by $-5a^2c$.

Ans. $-3b+2x^2-acd^2$.

5. Divide $25ab^3-15ab^2c^3-10a^2b$ by $-5ab$.

Ans.

6. Divide $6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3$ by $3a^2x^2y^3$.

Ans.

7. Divide $27a^5x^2+24a^4x^3-9a^3x^4$ by $3ax^2$.

Ans.

8. Divide $12a^4y^6-16a^5y^5+20a^6y^4-28a^7y^3$ by $-4a^4y^3$.

Ans.

CASE III.

(75.) *When the divisor and dividend are both polynomials.*

Let it be required to divide $a^2+2ab+b^2$ by $a+b$.

The object of this operation is to find a third polynomial which, multiplied by the second, will reproduce the first.

It is evident that the dividend is composed of all the partial products arising from the multiplication of each term of the divisor by each term of the quotient, these products being added together and reduced. Hence, if we divide the first term a^2 of the dividend by the first term a of the divisor, we shall obtain a term of the quotient, which is a. Multiplying each term of the divisor by a, and subtracting the product a^2+ab from the proposed dividend, the remainder may be regarded as the product of the divisor by the remaining terms of the quotient. We shall then obtain another

---

QUEST.—What is the third case? How do we obtain the first term of the quotient?

term of the quotient by dividing the first term of the remainder ab by the first term of the divisor a, which gives b. Multiplying the divisor by b, and subtracting as before, we find nothing remains. Hence $a+b$ is the exact quotient.

The operation may be exhibited as follows:

The dividend is	$a^2+2ab+b^2$	$a+b$ is the divisor.
	a^2+ab	$a+b$ is the quotient.
	$ab+b^2$	is the first remainder.
	$ab+b^2$	
	0	

It is generally convenient in algebra to place the divisor on the right of the dividend, and the quotient directly under the divisor.

In this example we have arranged the terms of the divisor and dividend *in the order of the powers of the letter* a. When the terms are thus arranged, we shall always obtain a term of the quotient by dividing the first term on the left of the dividend by the first term on the left of the divisor. Therefore, before commencing the division, the terms should be arranged in the order of the powers of one of the letters. Hence we deduce the following

(76.) RULE FOR THE DIVISION OF POLYNOMIALS.

1. *Arrange the terms of the dividend and divisor in the order of the powers of one of the letters.*

2. *Divide the first term on the left of the dividend by the first term on the left of the divisor; the result will be the first term of the quotient.*

QUEST.—In what order must we arrange the terms? Give the rule for the division of polynomials.

3. *Multiply the divisor by this term, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor; the result will be the second term of the quotient.*

5. *Multiply the divisor by this term, and subtract the product from the last remainder. Continue the same operation till all the terms of the dividend are exhausted.*

If the divisor is not exactly contained in the dividend, the quantity which remains after the division is finished must be placed over the divisor in the form of a fraction, and annexed to the quotient.

Ex. 2. Divide $x^3-a^3+3a^2x-3ax^2$ by $x-a$.

We here arrange the letters in the order of the powers of x.

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
x^3-3ax^2+3a^2x-a^3 & x-a \\ \hline
x^3-ax^2 & x^2-2ax+a^2 \\ \hline
-2ax^2+3a^2x-a^3 & \textit{Quotient.} \\
-2ax^2+2a^2x & \\ \hline
a^2x-a^3 & \\
a^2x-a^3 & \\ \hline
\end{array}$$

Ex. 3. Divide $a^3+2a^2b+2ab^2+b^3$ by a^2+ab+b^2.

Ans. $a+b$.

Ex. 4. Divide $24a^3-42a^2x+19ax^2-15x^3$ by $2a-3x$.

Ans. $12a^2-3ax+5x^2$.

Ex. 5. Divide $6a^3+17a^2b+24ab^2+16b^3$ by $3a+4b$.

Ans. $2a^2+3ab+4b^2$.

QUEST.—If there is a remainder, what must be done with it?

Ex. 6. Divide $ax^7+3a^2x^4-axy^6-3a^2xy^3$ by ax^3-ay^3.

Ans. x^4+xy^3+3ax.

Ex. 7. Divide $2a^4-5a^2b+2a^2c^2+3b^2-2bc^2$ by a^2-b.

Ans. $2a^2-3b+2c^2$.

Ex. 8. Divide $a^6+2a^3z^3+z^6$ by a^2-az+z^2.

Ans. $a^4+a^3z+az^3+z^4$.

Ex. 9. Divide $a^6-16a^3x^3+64x^6$ by $a^2-4ax+4x^2$.

Ans.

Ex. 10. Divide $a^4+6a^2x^2-4a^3x+x^4-4ax^3$

by $a^2-2ax+x^2$.

Ans. $a^2-2ax+x^2$.

Ex. 11. Divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

Ans.

Ex. 12. Divide $12x^4-192$ by $3x-6$.

Ans. $4x^3+8x^2+16x+32$

Ex. 13. Divide $a^6+3a^2b^4-3a^4b^2-b^6$

by $a^3-3a^2b+3ab^2-b^3$.

Ans. $a^3+3a^2b+3ab^2+b^3$.

Ex. 14. Divide $2a^2x^2+a^2bx+acx-6a^2b^2-5abc-c^2$

by $ax+2ab+c$.

Ans. $2ax-3ab-c$.

Ex. 15. Divide $a^4+2ab^3-2a^3b-4b^4$ by $a-2b$.

Ans. a^3+2b^3.

Ex. 16. Divide $4x^4-5x^2+12x-3$ by $2x^2+3x-1$.

Ans. $2x^2-3x+3$.

Ex. 17. Divide $2y^3-19y^2+26y-16$ by $2y^2-3y+2$.

Ans. $y-8$.

Ex. 18. Divide $a^6-2a^3x^3-2a^4x+4x^4$ by a^2-2x.

Ans. a^4-2x^3.

Ex. 19. Divide a^3-b^3 by $a-b$. *Ans.*

Ex. 20. Divide a^4-b^4 by $a-b$. *Ans.*

(77.) If the first term of the arranged dividend is not divisible by the first term of the arranged divisor, the complete division is impossible. So, also, the complete division is impossible when the first term of one of the remainders is not divisible by the first term of the divisor.

QUEST.—When will the complete division be impossible?

SECTION VI.

FRACTIONS.

(78.) WHEN a quotient is expressed as described in Art. 7, by placing the divisor under the dividend with a line between them, it is called a *fraction;* the dividend is called the numerator, and the divisor the denominator of the fraction. Algebraic fractions do not differ essentially from arithmetical fractions, and the same principles are applicable to both. The denominator shows into how many parts a unit is divided; and the numerator shows how many of those parts are used; or the denominator shows into how many parts the numerator is divided.

Thus, the fraction $\frac{6}{11}$ indicates that a unit has been divided into eleven equal parts, and that six of these parts are supposed to be taken.

So, also, the fraction $\frac{a}{b}$ indicates that a unit has been divided into b equal parts, and that a parts are supposed to be taken; or the numerator a is to be divided into b parts.

Every quantity which is not expressed under a fractional form is called an *entire* quantity.

An algebraic expression composed partly of an entire

QUEST.—What is a fraction? What does the denominator show? What does the numerator show? What is an entire quantity?

quantity and partly of a fraction, is called a *mixed* quantity.

Problem I.

(79.) *To reduce a fraction to lower terms.*

The fraction $\frac{1}{2}$ is evidently equal to $\frac{2}{4}$, or $\frac{3}{6}$, or $\frac{4}{8}$, or $\frac{5}{10}$, etc.

So, also, the fraction $\frac{20}{60}$ is equal to $\frac{10}{30}$, or $\frac{5}{15}$, or $\frac{1}{3}$.

In like manner, the fraction $\frac{a}{b}$ is equal to $\frac{2a}{2b}$, or $\frac{3a}{3b}$, or $\frac{10a}{10b}$, etc.

That is to say,

The value of a fraction is not changed if we multiply or divide both numerator and denominator by the same number.

Thus, $\frac{ab}{a}=\frac{abx}{ax}=\frac{abxy}{axy}=b.$

Hence, to reduce a fraction to lower terms, we have the following

Rule.

Divide both numerator and denominator by any quantity which will divide them both without a remainder.

If the numerator and denominator are both divided

Quest.—What is a mixed quantity? How may we reduce a fraction to lower terms? Give the rule.

until they no longer have any common factor, it is evident that the fraction will be reduced to its *lowest* terms. In the case of monomials, it is easy to detect the presence of a common factor; in the case of polynomials, they may often be detected by applying the principle of Art. 68.

Ex. 1. Reduce $\frac{4ax}{6bx}$ to its lowest terms.

Ans. $\frac{2a}{3b}$.

Ex. 2. Reduce $\frac{4ax^2}{5a^2bx^2}$ to its lowest terms.

Ans. $\frac{4}{5ab}$.

Ex. 3. Reduce $\frac{6adx^2}{24x^3}$ to its lowest terms.

Ans. $\frac{ad}{4x}$.

Ex. 4. Reduce $\frac{24abx^6}{12a^2x^4}$ to its lowest terms.

Ans. $\frac{2bx^2}{a}$.

Ex. 5. Reduce $\frac{36a^2b^4cx^5}{9a^3bx^2}$ to its lowest terms.

Ans. $\frac{4b^3cx^3}{a}$.

Ex. 6. Reduce $\frac{60a^4b^2cd^3}{48a^3b^4c^2d}$ to its lowest terms.

Ans. $\frac{5ad^2}{4bc}$.

QUEST.—How may we reduce a fraction to its lowest terms?

Ex. 7. Reduce $\frac{75ab^4mx}{21a^3b^2x}$ to its lowest terms.

Ans. $\frac{25b^2m}{7a^2}$.

Ex. 8. Reduce $\frac{15ax^2y^3}{45a^2x^3y^3}$ to its lowest terms.

Ans. $\frac{1}{3ax}$.

Ex. 9. Reduce $\frac{50a^4b^3mn}{25a^3b^2m^2n^2}$ to its lowest terms.

Ans. $\frac{2ab}{mn}$.

Ex. 10. Reduce $\frac{20a^5b^3c^3m}{28a^4bm^2}$ to its lowest terms.

Ans. $\frac{5ab^2c^3}{7m}$.

Ex. 11. Reduce $\frac{48m^2n^2x^2}{36mnxy}$ to its lowest terms.

Ans. $\frac{4mnx}{3y}$.

Ex. 12. Reduce $\frac{100a^3b^3c^3}{12ab^2cx^3}$ to its lowest terms.

Ans. $\frac{25a^2bc^2}{3x^3}$.

Ex. 13. Reduce $\frac{234a^3xy}{68a^2x^2y^2}$ to its lowest terms.

Ans. $\frac{117a}{34xy}$.

Ex. 14. Reduce $\frac{225a^3m^2x^2y^3}{60a^4mxy^2}$ to its lowest terms.

Ans. $\frac{15mxy}{4a}$.

Ex. 15. Reduce $\frac{4a^2+6a^4}{10a^2b^2+8a^3c}$ to its lowest terms.

Ans. $\frac{2+3a^2}{5b^2+4ac}$.

Ex. 16. Reduce $\frac{12a^4b+27a^3c}{39a^2}$ to its lowest terms.

Ans. $\frac{4a^2b+9ac}{13}$

Ex. 17. Reduce $\frac{21a^4b^2-35a^2b^4}{14a^2b^3+56a^3b^2}$ to its lowest terms.

Ans. $\frac{3a^2-5b^2}{2b+8a}$.

Ex. 18. Reduce $\frac{25a^2b^2x-30a^2b^2y}{35a^4b^2-45a^2b^3}$ to its lowest terms.

Ans. $\frac{5x-6y}{7a^2-9b}$.

Ex. 19. Reduce $\frac{a^2-b^2}{a^2+2ab+b^2}$ to its lowest terms.

This fraction may be written $\frac{(a+b)(a-b)}{(a+b)^2}$.
Rejecting the common factor $a+b$, we obtain

$$\frac{a-b}{a+b} \quad \textit{Ans.}$$

Ex. 20. Reduce $\frac{a^2-2ab+b^2}{a^2-b^2}$ to its lowest terms.

This fraction may be written $\frac{(a-b)^2}{(a+b)(a-b)}$.
Rejecting the common factor $a-b$, we obtain

$$\frac{a-b}{a+b} \quad \textit{Ans.}$$

PROBLEM II.

(80.) *To reduce a fraction to an entire or mixed quantity.*

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

Thus, $\frac{27}{5}$ is equal to $27\div5$, which equals $5\frac{2}{5}$.

Also, $\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x}$.

Examples.

1. Reduce $\frac{6678}{7}$ to an entire quantity.

Ans. 954.

2. Reduce $\frac{ax-x^2}{x}$ to an entire quantity.

Ans. $a-x$.

3. Reduce $\frac{ab-2a^2}{b}$ to a mixed quantity.

Ans. $a-\frac{2a^2}{b}$.

4. Reduce $\frac{x^2-y^2}{x-y}$ to an entire quantity.

Ans. $x+y$.

5. Reduce $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

Ans. $2x-1+\frac{3}{5x}$.

QUEST.—How may we reduce a fraction to an entire or mixed quantity?

6. Reduce $\dfrac{8b^3-16b+7a^2b^2}{8b}$ to a mixed quantity.

Ans. $b^2-2+\dfrac{7a^2b}{8}$.

7. Reduce $\dfrac{22a^3c^2-33a^4c^4+7ab}{11a^2c}$ to a mixed quantity

Ans. $2ac-3a^2c^3+\dfrac{7b}{11ac}$.

8. Reduce $\dfrac{45a^5x^2y^3+30a^4xy^2-8axy}{15axy^2}$ to a mixed quantity.

Ans. $3a^4xy+2a^3-\dfrac{8}{15y}$.

Problem III.

(81.) *To reduce a mixed quantity to the form of a fraction.*

Rule.

Multiply the entire part by the denominator of the fraction; to the product add the numerator, with its proper sign, and place the result over the given denominator.

Thus, $3\frac{2}{5}$ is equal to $\dfrac{3\times5+2}{5}=\dfrac{15+2}{5}=\dfrac{17}{5}$.

This result may be proved by the preceding rule. For $\dfrac{17}{5}$ is equal to $17\div5=3\frac{2}{5}$.

Also, $a+\dfrac{b}{c}$ is equal to $\dfrac{a\times c+b}{c}=\dfrac{ac+b}{c}$.

Quest.—How may we reduce a mixed quantity to the form of a fraction?

Examples.

1. Reduce $7\frac{5}{6}$ to the form of a fraction.

Ans. $\frac{47}{6}$.

2. Reduce $x+\frac{a^2-x^2}{x}$ to the form of a fraction.

Ans. $\frac{a^2}{x}$.

3. Reduce $x+\frac{ax+x^2}{2a}$ to the form of a fraction

Ans. $\frac{3ax+x^2}{2a}$.

4. Reduce $5+\frac{2x-7}{3x}$ to the form of a fraction.

Ans. $\frac{17x-7}{3x}$.

5. Reduce $1+\frac{x-a-1}{a}$ to the form of a fraction.

Ans. $\frac{x-1}{a}$.

6. Reduce $1+2x+\frac{x-3}{5x}$ to the form of a fraction.

Ans. $\frac{10x^2+6x-3}{5x}$.

7. Reduce $a+b+\frac{a-b}{x}$ to the form of a fraction.

Ans. $\frac{ax+bx+a-b}{x}$.

8. Reduce $2ab-3c+\frac{4ax}{5y}$ to the form of a fraction.

Ans. $\frac{10aby-15cy+4ax}{5y}$.

9. Reduce $11ac-4x+\frac{7ax-3b}{2am}$ to the form of a fraction.

Ans. $\frac{22a^2cm-8amx+7ax-3b}{2am}$.

10. Reduce $7+\frac{3b^2-8c^2}{a^2-b^2}$ to the form of a fraction

Ans. $\frac{7a^2-4b^2-8c^2}{a^2-b^2}$.

PROBLEM IV.

(82.) *To reduce fractions to a common denominator.*

RULE.

Multiply each numerator into all the denominators, except its own, for a new numerator, and all the denominators together for a common denominator.

Examples.

1. Reduce $\frac{3}{5}$ and $\frac{2}{7}$ to a common denominator.

We have seen in Art. 79, that if both numerator and denominator are multiplied by the same number, the value of the fraction will not be altered. If we multiply both the numerator and denominator of the first fraction by 7, and those of the second by 5, the frac-

QUEST.—How may we reduce fractions to a common denominator?

tions become $\frac{21}{35}$ and $\frac{10}{35}$,

where both fractions have the same denominator.

2. Reduce $\frac{1}{3}$, $\frac{3}{5}$, and $\frac{5}{7}$ to a common denominator.

Proceeding according to the rule, the numerator of
the first fraction becomes $1\times5\times7= 35$,
the second " $3\times3\times7= 63$,
the third " $5\times3\times5= 75$,
and the common denominator becomes $3\times5\times7=105$.

Hence the required fractions are $\frac{35}{105}$, $\frac{63}{105}$, and $\frac{75}{105}$,

which are evidently equivalent to the fractions proposed.

3. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to fractions having a common denominator. *Ans.* $\frac{ad}{bd}$, $\frac{bc}{bd}$.

Here it will be seen that the numerator and denominator of the first fraction are both multiplied by d, and in the second fraction they are both multiplied by b. The *value* of the fractions, therefore, is not changed by this operation.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator.

Ans. $\frac{ac}{bc}$, $\frac{ab+b^2}{bc}$.

QUEST.—How does it appear that the value of fractions is not changed by reducing them to a common denominator?

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, $\frac{6acd}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{4x}{5}$ to fractions having a common denominator.

Ans. $\frac{45}{60}$, $\frac{40x}{60}$, $\frac{60a+48x}{60}$.

7. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a-x}{a+x}$ to fractions having a common denominator.

Ans. $\frac{7a^2+7ax}{14a+14x}$, $\frac{6ax+6x^2}{14a+14x}$, $\frac{14a-14x}{14a+14x}$.

8. Reduce $\frac{x}{3}$, $\frac{x}{5}$, and $\frac{1-x}{1+x}$ to fractions having a common denominator.

Ans. $\frac{5x+5x^2}{15+15x}$, $\frac{3x+3x^2}{15+15x}$, $\frac{15-15x}{15+15x}$.

9. Reduce $\frac{a}{2b}$, $\frac{3ax}{5b}$, and $\frac{a^2-b^2}{c}$ to fractions having a common denominator.

Ans. $\frac{5abc}{10b^2c}$, $\frac{6abcx}{10b^2c}$, $\frac{10a^2b^2-10b^4}{10b^2c}$.

10. Reduce $\frac{a}{5}$, $\frac{b^2}{4}$, and $\frac{a^2+b^2}{a+b}$ to fractions having a common denominator.

Ans. $\frac{4a^2+4ab}{20a+20b}$, $\frac{5ab^2+5b^3}{20a+20b}$, $\frac{20a^2+20b^2}{20a+20b}$.

PROBLEM V.

(83.) *To add fractional quantities together.*

RULE.

Reduce the fractions, if necessary, to a common denominator; add the numerators together, and place their sum over the common denominator.

The fractions must first be reduced to a common denominator to render them like parts of unity. Before this reduction, they must be considered as unlike quantities.

Examples.

1. Add together $\frac{3}{7}$ and $\frac{2}{7}$.

Ans. $\frac{3+2}{7}=\frac{5}{7}$.

2. Add together $\frac{3}{5}$ and $\frac{4}{7}$.

These fractions, reduced to a common denominator, are $\frac{21}{35}$ and $\frac{20}{35}$. Hence their sum is $\frac{41}{35}$, or $1\frac{6}{35}$.

3. Add together $\frac{2a}{3bc}$ and $\frac{4ab}{3bc}$.

Ans. $\frac{2a+4ab}{3bc}$.

4. Add together $\frac{x}{2}$ and $\frac{x}{3}$.

QUEST.—How do we add fractions together? Why must the fractions be reduced to a common denominator?

Reducing to a common denominator, the fractions become $\frac{3x}{6}$ and $\frac{2x}{6}$.

Adding the numerators, we obtain $\frac{5x}{6}$ *Ans.*

5. What is the sum of $\frac{a}{b}$ and $\frac{c}{d}$?

Ans. $\frac{ad+bc}{bd}$.

6. Required the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Ans. $\frac{adf+bcf+bde}{bdf}$.

7. Required the sum of $\frac{2a}{3bc}$ and $\frac{3c}{4d}$.

Ans. $\frac{8ad+9bc^2}{12bcd}$.

8. Required the sum of $3a+\frac{2x}{5}$ and $a+\frac{8x}{9}$.

Ans. $4a+\frac{58x}{45}$.

9. Required the sum of $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

Ans. $\frac{a^2+b^2}{a^2-b^2}$.

10. Required the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

Ans. a.

11. Required the sum of $\frac{a}{2}$, $\frac{a-2m}{4}$, and $\frac{a+2m}{4}$.

Ans. a.

12. Required the sum of $\frac{ma-b}{m+n}$ and $\frac{na+b}{m+n}$.

Ans. a.

Problem VI.

(84.) *To subtract one fractional quantity from another.*

Rule.

Reduce the fractions to a common denominator; subtract one numerator from the other, and place their difference over the common denominator.

Examples.

1. From $\frac{5}{7}$ subtract $\frac{3}{5}$.

Reducing to a common denominator, the fractions become $\frac{25}{35}$ and $\frac{21}{35}$.

Hence, $\frac{25}{35}-\frac{21}{35}=\frac{4}{35}$ *Ans.*

2. From $\frac{2x}{3}$ subtract $\frac{3x}{5}$.

Reducing to a common denominator, the fractions become $\frac{10x}{15}$ and $\frac{9x}{15}$.

Hence, $\frac{10x}{15}-\frac{9x}{15}=\frac{x}{15}$ *Ans.*

3. From $\frac{12x}{7}$ subtract $\frac{3x}{5}$.

Ans. $\frac{60x-21x}{35}=\frac{39x}{35}$.

Quest.—How do we subtract one fraction from another?

4. From $\frac{ax}{b-c}$ subtract $\frac{ax}{b+c}$.

Ans. $\frac{2acx}{b^2-c^2}$.

5. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$.

Ans. b.

6. From $\frac{a+3b}{4x}$ subtract $\frac{2b+5}{7}$.

Ans. $\frac{7a+21b-8bx-20x}{28x}$.

7. From $\frac{a+b}{a}$ subtract $\frac{c+d}{d}$.

Ans. $\frac{ad+bd-ac-ad}{ad}$.

8. From $4a+\frac{b}{c}$ subtract $3a+\frac{d}{e}$.

Ans. $a+\frac{be-cd}{ce}$.

9. From $\frac{2ax-7x}{2mx}$ subtract $\frac{3ax}{4m}$.

Ans. $\frac{4ax-14x-3ax^2}{4mx}$.

10. From $11ab+2bc$ subtract $\frac{2ab}{3bm}$.

Ans. $\frac{33ab^2m+6b^2cm-2ab}{3bm}$.

PROBLEM VII.

(85.) *To multiply fractional quantities together.*

QUEST.—How do we multiply fractional quantities together?

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Before demonstrating this rule, we must establish the two following principles:

1. *In order to multiply a fraction by any number, we must multiply its numerator or divide its denominator by that number.*

Thus, the value of the fraction $\frac{ab}{a}$ is b. If we multiply the numerator by a, we obtain $\frac{a^2b}{a}$, or ab; and if we divide the denominator of the same fraction by a, we also obtain ab; that is, the original value of the fraction has been multiplied by a.

2. *In order to divide a fraction by any number, we must divide its numerator or multiply its denominator by that number.*

Thus, the value of the fraction $\frac{a^2b}{a}$ is ab. If we divide the numerator by a, we obtain $\frac{ab}{a}$, or b; and if we multiply the denominator of the same fraction by a, we obtain $\frac{a^2b}{a^2}$, or b; that is, the original value of the fraction ab has been divided by a.

Let it now be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

QUEST.—Upon what principles does this rule depend? How do we multiply a fraction by any number? How do we divide a fraction by any number?

First, let us multiply $\frac{a}{b}$ by c. According to the first of the preceding principles, the product must be $\frac{ac}{b}$.

But the proposed multiplier was $\frac{c}{d}$; that is, we have used a multiplier d times too great. We must therefore divide the result $\frac{ac}{b}$ by d; and according to the second of the preceding principles, we obtain

$$\frac{ac}{bd},$$

which result is the same as would have been obtained by multiplying together the numerators of the two fractions for a new numerator, and their denominators for a new denominator.

If the quantities to be multiplied are mixed, they must first be reduced to fractional forms.

Examples.

1. Multiply $\frac{3}{7}$ by $\frac{5}{8}$. *Ans.* $\frac{15}{56}$.

2. Multiply $\frac{x}{6}$ by $\frac{2x}{9}$. *Ans.* $\frac{x^2}{27}$.

3. Multiply $\frac{4x}{7}$ by $\frac{2a}{b}$. *Ans.* $\frac{8ax}{7b}$.

4. What is the continued product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$? *Ans.* $\frac{4x^3}{21}$.

5. Multiply $\frac{x}{a}$ by $\frac{x+a}{a+c}$. *Ans.* $\frac{x^2+ax}{a^2+ac}$.

6. What is the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$? *Ans.* $9ax$.

7. Multiply $\frac{3ab}{4cd}$ by $\frac{4am}{5bn}$. *Ans.* $\frac{3a^2bm}{5bcdn}$.

8. Multiply $\frac{9anx}{13bmy}$ by $\frac{3ax^2}{4by^2}$. *Ans.* $\frac{27a^2nx^3}{52b^2my^3}$.

9. Multiply $\frac{4a}{b+3c}$ by $\frac{2ab-3x}{4ax}$. *Ans.* $\frac{2ab-3x}{bx+3cx}$.

10. Multiply $b+\frac{bx}{a}$ by $\frac{a}{x}$. *Ans.* $\frac{ab+bx}{x}$.

11. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

12. What is the continued product of x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$? *Ans.* $\frac{x^3-x}{a^2+ab}$.

Problem VIII.

(86.) *To divide one fractional quantity by another.*

Rule.

Invert the divisor, and proceed as in multiplication.

If the two fractions have the same denominator, then the quotient of the fractions will be the same as the quotient of their numerators.

Quest.—How do we divide one fraction by another? Explain the reason of the rule.

Thus it is plain that $\frac{3}{11}$ is contained in $\frac{9}{11}$ as often as 3 is contained in 9.

But when the two fractions have not tne same denominator, we must reduce them to this form by Problem IV.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

Reducing the fractions to a common denominator, we have $\frac{ad}{bd}$ to be divided by $\frac{bc}{bd}$.

It is now plain that the quotient must be represented by the division of ad by bc, which gives

$$\frac{ad}{bc},$$

the same result as obtained by the above rule.

Thus, $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

Examples.

1. Divide $\frac{5}{7}$ by $\frac{3}{11}$. *Ans.* $\frac{55}{21}$ or $2\frac{13}{21}$.

2. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

3. Divide $\frac{2a}{b}$ by $\frac{4c}{d}$. *Ans.* $\frac{ad}{2bc}$.

4. Divide $\frac{3x}{5}$ by $\frac{4x}{7}$. *Ans.* $\frac{21}{20}$.

5. Divide $\frac{4x^2}{11}$ by $\frac{5x}{3}$. *Ans.* $\frac{12x}{55}$.

6. Divide $\frac{a+b}{14}$ by $\frac{2a}{7}$. *Ans.* $\frac{a+b}{4a}$.

7. Divide $\frac{15}{a-b}$ by $\frac{5}{b}$. *Ans.* $\frac{3b}{a-b}$.

8. Divide $\frac{a-x}{6bd}$ by $\frac{7mn}{3b}$. *Ans.* $\frac{a-x}{14dmn}$.

9. Divide $\frac{2x^2}{a^2+x^2}$ by $\frac{x}{a+x}$. *Ans.* $\frac{2ax+x^2}{a^2+x^2}$.

10. Divide $\frac{x+1}{6}$ by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

11. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

12. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$. *Ans.* $\frac{2a+x}{c^2+cx+x^2}$.

SECTION VII.

EQUATIONS OF THE FIRST DEGREE.

(87.) ***An equation is a proposition which declares the equality of two quantities expressed algebraically.***

Thus, $x-4=a-x$ is a proposition expressing the equality of the quantities $x-4$ and $a-x$.

(88.) Equations are usually composed of certain quantities which are *known*, and others which are *unknown*. The known quantities are represented either by numbers or by the first letters of the alphabet, a, b, c, etc.; the unknown quantities by the last letters, x, y, z, etc.

(89.) A *root* of an equation is the value of the unknown quantity in the equation.

(90.) Equations are divided into *degrees*, according to the highest power of the unknown quantity which they contain.

Those which contain only the *first* power of the unknown quantity are called equations of the *first degree*.

As $$ax+b=cx+d.$$

Those in which the highest power of the unknown quantity is a *square*, are called equations of the *second degree*, etc.

Quest.—What is an equation? How are known and unknown quantities represented? What is a root of an equation? What is an equation of the first degree? What is an equation of the second degree?

Thus $3x^2-2x=40$ is an equation of the second degree.

(91.) To *solve* an equation is to find the value of the unknown quantity; or to find a number which, substituted for the unknown quantity in the equation, proves the two members of the equation to be equal to each other.

(92.) The following principles are regarded as self-evident, and are called *axioms:*

1. If equal quantities be *added* to both members of an equation, the equality of the members will not be destroyed.

2. If equal quantities be *subtracted* from both members of an equation, the equality will not be destroyed.

3. If both members of an equation be *multiplied* by the same number, the equality will not be destroyed.

4. If both members of an equation be *divided* by the same number, the equality will not be destroyed.

(93.) The unknown quantity may be combined with the known quantities in the given equation by the operations of *addition*, *subtraction*, *multiplication*, or *division*.

We shall consider these different cases in succession.

I. The unknown quantity may be combined with known quantities by *addition*.

Let it be required to solve the equation

$$x+5=25.$$

If from the two equal quantities $x+5$ and 25 we subtract the same quantity 5, the remainders will be

QUEST.—What is meant by solving an equation? Name the axioms employed in algebra. When known quantities are added to the unknown quantity, how do we solve the equation?

equal, according to the last Article, and we shall have

$$x+5-5=25-5,$$

or $\quad x=20$, the value required.

So, also, in the equation

$$x+a=b,$$

subtracting a from each of the equal quantities $x+a$ and b, the result is

$$x=b-a, \text{ the value required.}$$

(94.) II. The unknown quantity may be combined with known quantities by *subtraction.*

Let the equation be

$$x-5=15.$$

If to the two equal quantities $x-5$ and 15, the same quantity 5 be added, the sums will be equal according to Art. 92, and we have

$$x-5+5=15+5,$$

or $\quad x=20$, the value required.

So, also, in the equation

$$x-a=b,$$

adding a to each of these equal quantities, the result is $\quad x=b+a$, the value required.

From the preceding examples we conclude that

We may transpose any term of an equation from one member to the other by changing its sign.

We may change the sign of every term of an equation without destroying the equality.

This is, in fact, the same thing as transposing *every term* in each member of the equation.

QUEST.—When known quantities are subtracted from the unknown quantity, how do we solve the equation? How may the terms of an equation be transposed? What change may be made in the signs of the terms?

If the same quantity appear in each member of an equation affected with the same sign, it may be suppressed.

(95.) III. The unknown quantity may be combined with known quantities by *multiplication.*

Let the equation be

$$5x=25.$$

If we divide each of the equal quantities $5x$ and 25 by the same quantity 5, the quotients will be equal, and we shall have

$$x=5, \text{ the value required.}$$

So, also, in the equation

$$ax=b,$$

dividing each of these equals by a, the result is

$$x=\frac{b}{a}, \text{ the value required.}$$

From this we conclude that

When the unknown quantity is multiplied by a known quantity, the equation is solved by dividing both members by this known quantity.

(96.) IV. The unknown quantity may be combined with known quantities by *division.*

Let the equation be

$$\frac{x}{5}=4.$$

If we multiply each of the equal quantities $\frac{x}{5}$ and 4 by the same quantity 5, the products will be equal, and we shall have

$$x=20, \text{ the value required.}$$

QUEST.—When may a quantity be suppressed? When the unknown quantity is multiplied by a known quantity, how is the equation solved?

So, also, in the equation

$$\frac{x}{a}=b,$$

multiplying each of these equals by a, the result is

$x=ab$, the value required.

From this we conclude that

When the unknown quantity is divided by a known quantity, the equation is solved by multiplying both members by this known quantity.

(97.) V. Several terms of an equation may be *fractional.*

Let the equation be

$$\frac{x}{2}=\frac{2}{3}+\frac{4}{5}.$$

Multiplying each of these equals by 2, the result is

$$x=\frac{4}{3}+\frac{8}{5}.$$

Multiplying each of these last equals by 3, we obtain

$$3x=4+\frac{24}{5};$$

and multiplying again by 5, we obtain

$$15x=20+24,$$

an equation free from fractions.

We might have obtained the same result by multiplying the original equation at once by the product of all the denominators.

Thus, multiplying by $2\times3\times5$, we have

$$\frac{30x}{2}=\frac{60}{3}+\frac{120}{5};$$

QUEST.—When the unknown quantity is divided by a known quantity, how is the equation solved? When several terms of an equation are fractional, how should we proceed?

or, reducing, we have

$$15x=20+24, \text{ as before.}$$

So, also, in the equation

$$\frac{x}{a}=\frac{b}{c}+\frac{d}{e},$$

multiplying successively by all the denominators, or by a c, e at once, we obtain

$$\frac{acex}{a}=\frac{abce}{c}+\frac{acde}{e}.$$

Canceling from each term the letter which is common to its numerator and denominator, we have

$$cex=abe+acd,$$

an equation clear of fractions.

Hence it appears that

An equation may be cleared of fractions by multiplying each member into all the denominators.

We will now apply these principles to the solution of equations.

Ex. 1. Given $5x+8=4x+10$ to find the value of x.

Transposing $4x$ to the first member of the equation, and 8 to the second member, taking care to change their signs (Art. 94), we have

$$5x-4x=10-8.$$

Uniting similar terms, we find

$$x=2.$$

In order to verify this result, put 2 in the place of x wherever it occurs in the original equation, and we shall obtain

$$5\times2+8=4\times2+10.$$

That is, $10+8=8+10,$

or, $18=18,$

which proves that we have found the correct value of x

Ex. 2. Given $x-7=\frac{x}{5}+\frac{x}{3}$, to find the value of x.

Multiplying every term of the equation by 5, and also by 3, in order to clear it of fractions (Art. 97), we obtain

$$15x-105=3x+5x.$$

Hence, by transposition, Art. 94,

$$15x-3x-5x=105,$$

or, $$7x=105,$$

and therefore $$x=\frac{105}{7}=15.$$

To verify this result, put 15 in the place of x in the original equation, and we have

$$15-7=\frac{15}{5}+\frac{15}{3}.$$

That is, $$15-7=3+5,$$

or, $$8=8,$$

which shows that we have found the correct value of x.

Ex. 3. Given $3ax-4ab=2ax-6ac$, to find the value of x in terms of b and c.

Dividing every term by a, we have

$$3x-4b=2x-6c.$$

By transposition,

$$3x-2x=4b-6c,$$

or, $$x=4b-6c.$$

This result may be verified in the same manner as the preceding.

(98.) Hence, in order to solve an equation of the first degree containing one unknown quantity, we have the following

RULE.

1. *Clear the equation of fractions, and perform in both members all the algebraic operations indicated.*

2. *Transpose all the terms containing the unknown quantity to one side, and all the remaining terms to the other side of the equation, and reduce each member to its most simple form.*

3. *Divide each member by the coefficient of the unknown quantity.*

Ex. 4. Given $5x+16-7=29$ to find the value of x.

Ans. $x=4$.

Ex. 5. Given $x+12=4x-6$ to find the value of x.

Ans. $x=6$.

Ex. 6. Given $7-3x+12=25-4x-1$ to find the value of x.

Ans. $x=5$.

Ex. 7. Given $2x+\frac{x}{2}+\frac{x}{5}=27$ to find the value of x.

Ans. $x=10$.

Ex. 8. Given $3x-\frac{x}{3}+4=6x-6$ to find the value of x.

Ans. $x=3$.

Ex. 9. Given $\frac{x}{3}-\frac{2x}{5}+x=\frac{3x}{7}+53$ to find the value of x. *Ans.* $x=105$.

Ex. 10. Given $x+\frac{x}{3}+\frac{x}{5}-\frac{x}{6}=3x-49$ to find the value of x.

Ans. $x=30$.

QUEST.—Give the rule for solving an equation of the first degree with one unknown quantity.

Ex. 11. Given $2x+\frac{ax-b}{3}=x-a$ to find the value of x. *Ans.* $x=\frac{b-3a}{a+3}$.

Ex. 12. Given $\frac{x}{a}+\frac{x}{b}-\frac{x}{c}=d$ to find the value of x.

Ans. $x=\frac{abcd}{bc+ac-ab}$.

Ex. 13. Given $2ax+5-\frac{a}{3}=bx+a$ to find the value of x.

Ans. $x=\frac{4a-15}{6a-3b}$.

Ex. 14. Given $3x^2-10x=8x+x^2$ to find the value of x.

Ans. $x=9$.

Ex. 15. Given $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}$ to find the value of x.

Ans. $x=\frac{d}{c}$.

Ex. 16. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$ to find the value of x. *Ans.* $x=9$.

Ex. 17. Given $\frac{ab}{x}=bc+d+\frac{1}{x}$ to find the value of x.

Ans. $x=\frac{ab-1}{bc+d}$.

Ex. 18. Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$ to find the value of x. *Ans.* $x=7$.

Ex. 19. Given $5ax-2b+4bx=2x+5c$ to find the value of x.

$$Ans.\ x=\frac{5c+2b}{5a+4b-2}.$$

Ex. 20. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find the value of x.

Ans. $x=9$.

(99.) An equation may always be cleared of fractions by multiplying each member into *all* the denominators, according to Art. 97. But sometimes the same object may be attained by a less amount of multiplication.

Thus, in the preceding example, the equation may be cleared of fractions by multiplying each term by 16 instead of $16\times8\times2$, and it is important to avoid all useless multiplication. In general, it is sufficient to multiply by the *least common multiple* of all the denominators.

A common multiple of two or more numbers is any number which they will divide without a remainder; and the *least* common multiple is the least number which they will so divide. Thus 16 is the least common multiple of 16, 8, and 2.

Ex. 21. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find the value of x.

The minus sign prefixed to the fraction $\frac{2x-4}{3}$

QUEST.—How may an equation be cleared of fractions with the least amount of multiplication? What is a common multiple? What is the least common multiple of several numbers?

shows that the value of the entire fraction is to be subtracted from 12; and since a quotient is subtracted by changing its sign, the equation, cleared of fractions, will stand

$$6x+9x-15=72-4x+8,$$

which gives $x=5$ *Ans.*

Ex. 22 Given $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$ to find the value of x.

Ans. $x=7$.

Ex. 23. Given $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$ to find the value of x.

Ans. $x=4$.

Ex. 24. Given $\frac{x-2}{2}+\frac{x}{4}=15-\frac{4x-2}{3}$ to find the value of x.

Ans. $x=8$.

Ex. 25. Given $\frac{x+5}{2}+2=13-\frac{x-2}{3}$ to find the value of x.

Ans. $x=11$.

Ex. 26. Given $\frac{x}{5}-\frac{x-2}{3}+\frac{x}{4}=\frac{x}{2}-7$ to find the value of x. *Ans.* $x=20$.

Ex. 27. Given $\frac{x}{7}-\frac{x-4}{5}+\frac{x}{2}=x-7$ to find the value of x.

Ans. $x=14$.

QUEST.—What does a minus sign prefixed to a fraction show?

Ex. 28. Given $x-\frac{2x-2}{11}+\frac{3x-1}{7}=x+3$ to find the value of x.

Ans. $x=12$.

Ex. 29. Given $2x-\frac{3x-4}{5}=\frac{4x-14}{2}$ to find the value of x.

Ans. $x=13$.

Ex. 30. Given $3x-a+cx=\frac{a+x}{3}-\frac{b-x}{a}$ to find the value of x.

Ans. $x=\frac{4a^2-3b}{8a+3ac-3}$.

(100.) PROBLEMS PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING BUT ONE UNKNOWN QUANTITY.

It has already been observed, Art. 13, that the solution of a problem by algebra consists of two distinct parts:

1st. To express the conditions of the problem algebraically; that is, to *form* the equation.

2d. To *solve* the equation.

(101.) The second operation has been fully explained; but the first is often more embarrassing to beginners than the second. Sometimes the statement of a problem furnishes the equation directly; and sometimes it is necessary to deduce from the statement new conditions, which are to be expressed algebraically.

It is impossible to give a general rule which will enable us to translate every problem into algebraic

QUEST.—The solution of an algebraic problem consists of what parts? How do we reduce a problem to an equation?

language. The ability to do this with facility can only be acquired by reflection and practice.

(102.) In most cases, however, we may obtain the desired equation by applying the following

RULE.

Denote one of the required quantities by **x**; *and then indicate by means of the algebraic signs the same operations on the known and unknown quantities as would be necessary to verify the value of the unknown quantity, if it were already known.*

Problem 1. What number is that to the double of which, if 20 be added, the sum is equal to four times the required number?

Let x represent the number required.

The double of this will be $2x$.

This, increased by 20, should equal $4x$.

Hence, by the conditions, $2x+20=4x$.

The problem is now translated into algebraic language, and it only remains to solve the equation in the usual way.

Transposing, we obtain

$$20=4x-2x=2x,$$

and $$10=x,$$

or, $$x=10.$$

To verify this number, we have but to double 10, and add 20 to the result; the sum is 40, which is equal to four times 10, according to the conditions of the problem.

Prob. 2. What number is that the double of which exceeds its half by 9?

QUEST.—Give the rule for forming the equation.

Let $x=$ the number required.

Then, by the conditions,

$$2x-\frac{x}{2}=9.$$

Clearing of fractions,

$$4x-x=18,$$

or, $$3x=18.$$

Hence, $$x=6.$$

To verify this result, double 6, which makes 12, and diminish it by the half of 6 or 3; the result is 9, according to the conditions of the problem.

Prob. 3. The sum of two numbers is 14, and their difference is 4. What are those numbers?

Let $x=$ the least number.

Then $x+4$ will be the greater number.

The sum of these is $2x+4$, which is required to equal 14. Hence we have

$$2x+4=14.$$

By transposition,

$$2x=14-4=10,$$

and $x=5$, the least number.

Also, $x+4=9$, the greater number.

Verification. $\left.\begin{array}{l}9+5=14\\ 9-5=\ 4\end{array}\right\}$ according to the conditions.

The following is a generalization of the preceding problem:

Prob. 4. The sum of two numbers is a, and their difference b. What are those numbers?

Let x represent the least number.

Then $x+b$ will represent the greater number.

The sum of these is $2x+b$, which is required to equal a. Hence we have

$$2x+b=a.$$

By transposition,

$$2x=a-b,$$

or, $x=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}$, the less number.

Hence, $x+b=\frac{a}{2}-\frac{b}{2}+b=\frac{a}{2}+\frac{b}{2}$, the greater number.

(103.) As these results are independent of any particular value attributed to the letters a and b, it follows that

Half the difference of two quantities, added to half their sum, is equal to the greater; and

Half the difference subtracted from half the sum is equal to the less.

The expressions $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$ are called *formulas*, because they may be regarded as comprehending the solution of all questions of the *same kind;* that is, of all problems in which we have given the sum and difference of two quantities.

Thus, let $\left.\begin{array}{l} a=14 \\ b=\ 4 \end{array}\right\}$ as in the preceding problem.

Then $\frac{a+b}{2}=\frac{14+4}{2}=9$, the greater number,

and $\frac{a-b}{2}=\frac{14-4}{2}=5$, the less number.

So, also, we shall find, if

QUEST.—How may two quantities be determined from their sum and difference? What is a formula?

the sum of two numbers is	and their difference is	the numbers will be
10	6	8 and 2,
10	4	7 and 3,
14	6	10 and 4,
16	2	9 and 7.
25	5	
60	10	
32	8	
44	6	

Prob. 5. From two towns which are 63 miles distant, two travelers set out at the same time with an intention of meeting. One of them goes 4 miles, and the other 5 miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then $4x$ wil: represent the number of miles one traveled,
and $5x$ the number the other traveled;
and since they meet, they must together have traveled the whole distance.

Consequently, $4x+5x=63.$

Hence, $9x=63,$

or, $x=7.$

Proof. In 7 hours, at 4 miles an hour, one would travel 28 miles; the other, at 5 miles an hour, would travel 35 miles. The sum of 28 and 35 is 63 miles, which is the whole distance.

The following is a generalization of the preceding problem.

Prob. 6. From two points which are a miles apart, two bodies move toward each other, the one at the rate of m miles per hour, the other at the rate of n miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then mx will represent the number of miles one body moves,
and nx the miles the other body moves,
and we shall obviously have

$$mx+nx=a.$$

Hence, $$x=\frac{a}{m+n}.$$

This is a general formula, comprehending the solution of all problems of this kind. Thus:

Let the distance be	Let one body move	and the other	miles per hour; the time of meeting will be
150 miles.	6	4	15 hrs
90 "	8	7	6 "
135 "	15	12	
210 "	20	15	
192 "	14	10	
540 "	25	20	

We see that an infinite number of problems may be proposed, all similar to Prob. 5, but they are all solved by the formula of Prob. 6.

Prob. 7. A bookseller sold 10 books at a certain price, and afterward 15 more at the same rate. Now at the last sale he received 25 dollars more than at the first. What did he receive for each book?

Ans. Five dollars.

The following is a generalization of the preceding problem.

Prob. 8. Find a number such that when multiplied successively by m and by n, the difference of the products shall be a.

Let x represent the required number.

Then, by the conditions of the problem,

$$mx-nx=a.$$

Hence, $$x=\frac{a}{m-n}.$$

This formula comprehends the solution of an infinite number of problems all similar to Prob. 7. Thus:

If a certain number be multiplied by	and also by	the difference of the products is	Hence the required number is
12	8	28	7
9	6	45	15
15	9	66	
11	5	120	
25	14	132	

Prob. 9. A gentleman dying bequeathed \$900 to three servants. A was to have twice as much as B, and B three times as much as C. What were their respective shares?

Ans. A received \$540, B \$270, and C \$90.

The following is a generalization of the preceding problem.

Prob. 10. Divide the number a into three such parts that the second may be m times as great as the first, and the third n times as great as the second.

Ans. $\frac{a}{1+m+mn}$, $\frac{ma}{1+m+mn}$, $\frac{mna}{1+m+mn}$.

Prob. 11. Four merchants entered into a speculation, for which they subscribed \$9510, of which B paid three times as much as A, C paid as much as A and B, and D paid as much as C and B. What did each pay?

Let $x=$ the number of dollars A paid.

Then we shall have

$$x+3x+4x+7x=9510;$$

whence $x=634$ dollars A paid;

B paid 1902 dollars;
C paid 2536 "
and D paid 4438 "

Prob. 12. A draper bought three pieces of cloth, which together measured 111 yards. The second piece was 11 yards longer than the first, and the third 17 yards longer than the second. What was the length of each?

Ans. 24, 35, and 52 yards respectively.

Prob. 13. A hogshead which held 92 gallons was filled with a mixture of brandy, wine, and water. There were 10 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Ans. Brandy 18 gallons, wine 28 gallons, and water 46 gallons.

The following is a generalization of the preceding problem.

Prob. 14. Divide the number a into three such parts that the second shall exceed the first by m, and the third shall be equal to the sum of the first and second.

Ans. $\frac{a-2m}{4}, \frac{a+2m}{4}, \frac{a}{2}.$

Prob. 15. A farmer employed four workmen, to the first of whom he gave 2 shillings more than to the second, to the second 3 shillings more than to the third, and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Ans. They received 12, 10, 7, and 3 shillings respectively.

Prob. 16. A father divided a certain sum of money among his four sons. The third had 11 shillings more than the fourth, the second 16 shillings more than the third, and the eldest 19 shillings more than the second; and the whole sum was 16 shillings more than 6 times the sum which the youngest received. How much had each?

Ans. 34, 45, 61, and 80 shillings respectively.

(104.) Problems which involve several unknown quantities may often be solved by the use of a single unknown letter. Most of the preceding examples are of this kind. In general, when we have given the sum or difference of two quantities, both of them may be expressed by means of the same letter. For the difference of two quantities added to the less must be equal to the greater; and if one of two quantities be subtracted from their sum, the remainder will be equal to the other.

Prob. 17. At a certain election 25,000 votes were polled, and the candidate chosen wanted but 2000 of having twice as many votes as his opponent. How many voted for each?

Let $x=$ the number of votes for the unsuccessful candidate.

Then $25{,}000-x=$ the number the successful one had,

and $$25{,}000-x+2000=2x.$$

Ans. 9000 and 16,000.

The following is a generalization of the preceding problem.

QUEST.—When we have given the sum and difference of two quantities, how may each of them be expressed?

Prob. 18. Divide the number a into two such parts that one part increased by b shall be equal to m times the other part.

Ans. $\frac{a+b}{m+1}, \frac{ma-b}{m+1}$.

Prob. 19. A train of cars moving at the rate of 20 miles per hour, had been gone three hours when a second train followed at the rate of 25 miles per hour. In what time will the second train overtake the first?

Let $x=$ the number of hours the second train is in motion,

and $x+3=$ the time of the first train.

Then $25x=$ the number of miles traveled by the second train,

and $20(x+3)=$ the miles traveled by the first train.

But at the time of meeting they must both have traveled the same distance.

Therefore, $25x=20x+60$.

By transposition,

$$5x=60,$$

and $$x=12.$$

Proof. In 12 hours, at 25 miles per hour, the second train goes 300 miles, and in 15 hours, at 20 miles per hour, the first train also goes 300 miles; that is, it is overtaken by the second train.

The following is a generalization of the preceding problem.

Prob. 20. Two bodies move in the same direction from two places at a distance of a miles apart, the one at the rate of n miles per hour, the other pursuing at the rate of m miles per hour. When will they meet?

Ans. In $\frac{a}{m-n}$ hours.

This problem, it will be seen, is essentially the same as Prob. 8.

Prob. 21. A vintner fills a cask containing 86 gallons with a mixture of brandy, wine, and water. There are 18 gallons of water more than of brandy, and 14 more of wine than of water. How many are there of each?

Ans. 12 gallons of brandy,
30 gallons of water,
44 gallons of wine.

Prob. 22. A gentleman gave 34 shillings to two poor persons; but he gave 4 shillings more to one than to the other. What did he give to each?

Ans. 15 and 19 shillings.

Prob. 23. What two numbers are those whose sum is 66 and difference 18?

Ans. 24 and 42.

Prob. 24. Two persons began to play with equal sums of money. The first lost 20 shillings, the other won 28 shillings; and then the second had twice as many shillings as the first. What sum had each at first? *Ans.* 68 shillings.

Prob. 25. A farmer has two flocks of sheep, each containing the same number. From one of them he sells 24, and from the other 129, and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

Ans. 234.

Prob. 26. Divide the number 181 into two such parts that four times the greater may exceed five times the less by 49.

Ans. 75 and 106.

Prob. 27. Divide the number a into two such parts that m times the greater may exceed n times the less by b.

Ans. $\frac{ma-b}{m+n}, \frac{na+b}{m+n}$.

Prob. 28. A prize of 2125 dollars was divided between two persons, A and B, whose shares were in the ratio of 5 to 12. What was the share of each?

Ans.

(105.) Beginners uniformly put x to represent one of the quantities sought in a problem; but a solution may often be very much simplified by pursuing a different method. Thus, in the preceding problem we may put x to represent one fifth of A's share. Then $5x$ will be A's share, and $12x$ will be B's, and we shall have the equation

$$5x+12x=2125,$$

and

$$x=125;$$

consequently, their shares were 625 and 1500 dollars.

The following is a generalization of the preceding problem.

Prob. 29. Divide the number a into two such parts that the first part may be to the second as m to n.

Ans. $\frac{ma}{m+n}, \frac{na}{m+n}$.

Prob. 30. Divide the number 73 into two such parts

QUEST.—Is it always best to represent the required quantity by x?

that the difference between the greater and 77 may equal three times the difference between the less and 40.

Ans. 29 and 44.

Prob. 31. What number is that whose third part exceeds its fourth part by 15?

Let $12x=$ the number.

Then $$4x-3x=15,$$

or, $$x=15.$$

Therefore the number $=12\times15=180$.

The following is a generalization of the preceding problem.

Prob. 32. Find a number such that when it is divided successively by m and by n, the difference of the quotients shall be a.

Ans. $\frac{mna}{n-m}$.

Prob. 33. What two numbers are as 2 to 3, to each of which, if four be added, the sums will be as 5 to 7?

Let $2x$ and $3x$ represent the required numbers.

Then $$2x+4:3x+4::5:7.$$

But when four quantities are proportional, *the product of the extremes is equal to the product of the means.* Hence

$$14x+28=15x+20.$$

Therefore, $$x=8.$$

And the required numbers are 16 and 24.

Prob. 34. A sum of money is to be divided between two persons, A and B, so that as often as A receives 9 pounds, B takes 4. Now it happens that A receives

QUEST.—How may a proportion be reduced to an equation?

20 pounds more than B. What are their respective shares?

Ans. A receives 36 pounds and B 16 pounds.

Prob. 35. A merchant bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger as in the other. How many were there in each at first?

Ans. 36 and 12 gallons respectively

Prob. 36. A gentleman divides two dollars among 12 children, giving to some 18 cents each, and to the rest 14 cents. How many were there of each class?

Ans. 8 of the first class and 4 of the second.

Prob. 37. A fish was caught whose tail weighed 9 pounds. His head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

Ans. 72 pounds.

Prob. 38. If the sun moves every day one degree, and the moon thirteen, and the sun is now 60 degrees in advance of the moon, when will they be in conjunction for the first time, second time, and so on?

Ans. In 5 days, 35 days, 65 days, etc.

Prob. 39. Divide the number 15 into two such parts that the difference of their squares may be 45.

Ans. 9 and 6.

The following is a generalization of the preceding problem.

Prob. 40. Divide the number a into two such parts that the difference of their squares may be b.

Ans. $\frac{a^2-b}{2a}, \frac{a^2+b}{2a}$.

Prob. 41. The estate of a bankrupt, valued at 21,000 dollars, is to be divided among three creditors according to their respective claims. The debts due to A and B are as 2 to 3, while B's claims and C's are in the ratio of 4 to 5. What sum must each receive?

Ans. A receives 4800 dollars;
B receives 7200 dollars;
C receives 9000 dollars.

Prob. 42. A grocer has two kinds of tea, one worth 72 cents per pound, the other 40 cents. How many pounds of each must be taken to form a chest of 48 pounds which shall be worth 60 cents?

Ans. 30 pounds at 72 cents, and 18 pounds at 40 cents.

Prob. 43. A can perform a piece of work in 6 days, B can perform the same work in 8 days, and C can perform the same work in 24 days. In what time will they finish it if all work together?

Ans. 3 days.

Prob. 44. There are three workmen, A, B, and C. A and B together can perform a piece of work in 27 days, A and C together in 36 days, and B and C together in 54 days. In what time could they finish it if all worked together?

A and B together can perform $\frac{1}{27}$ of the work in one day;

A and C together can perform $\frac{1}{36}$ of the work in one day;

B and C together can perform $\frac{1}{54}$ of the work in one day.

Therefore, adding these three quantities,

$2A+2B+2C$ can perform $\frac{1}{27}+\frac{1}{36}+\frac{1}{54}$ in one day.
$=\frac{1}{12}$ in one day.

Therefore, A, B, and C together can perform $\frac{1}{24}$ of the work in one day; that is, they can finish it in 24 days. If we put x to represent the time in which they would all finish it, then they would together perform $\frac{1}{x}$ part of the work in one day, and we should have

$$\frac{1}{27}+\frac{1}{36}+\frac{1}{54}=\frac{2}{x}.$$

From which equation we find $x=24$, *Ans.*

Prob. 45. Divide the number 45 into four such parts that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal.

In solving examples of this kind, several unknown quantities are usually introduced, but this practice is worse than superfluous. The four parts into which 45 is to be divided may be represented thus:

The first, $=x-2$;
second, $=x+2$;
third, $=\frac{x}{2}$;
fourth, $=2x$;

for if the first expression be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the result in each case will be x. The sum of the four parts is $4\frac{1}{2}x$, which must equal 45.

Hence, $x=10$.

Therefore, the parts are 8, 12, 5, and 20.

Prob. 46. A father, aged 54 years, has a son aged

9 years. In how many years will the age of the father be four times that of the son?

Ans. 6 years.

Prob. 47. The age of a father is represented by a, the age of his son by b. In how many years will the age of the father be n times that of the son?

Ans. $\frac{a-nb}{n-1}$ years.

Prob. 48. It is required to divide the number 36 into three such parts that one half of the first, one third of the second, and one fourth of the third may be equal to each other.

Ans. 8, 12, and 16.

Prob. 49. Divide the number 50 into two such parts that the greater increased by 5, may be to the less diminished by 5, as 7 to 3.

Ans. 30 and 20.

Prob. 50. What number is that to which, if 1, 4, and 10 be severally added, the first sum shall be to the second as the second to the third?

Ans. 2.

SECTION VIII.

EQUATIONS OF THE FIRST DEGREE CONTAINING TWO UNKNOWN QUANTITIES.

(106.) In the examples which have been hitherto given, each problem has contained but one unknown quantity; or, if there have been more, they have been so related to each other, that all have been expressed by means of the same letter. This, however, can not always be done, and we are now to consider how equations of this kind are resolved.

(107.) If we have *two* equations with two unknown quantities, we must endeavor to deduce from them *a single equation* containing only *one* unknown quantity. We must, therefore, make one of the unknown quantities disappear; or, as it is termed, we must *eliminate* it. There are three different methods of elimination which may be practiced.

The *first* is by substitution;
" *second* is by comparison;
" *third* is by addition and subtraction.

(108.) Elimination by Substitution.

Ex. 1. Let it be proposed to solve the system of equations

Quest.—When we have two equations with two unknown quantities, how must we proceed? What is meant by elimination? How many methods of elimination are practiced?

$$\left.\begin{aligned} x+y&=12 \\ x-y&=\ \ 6. \end{aligned}\right\}$$

From the second equation we find the value of x in terms of y, which gives

$$x=y+6.$$

Substituting the expression $y+6$ for x in the first equation, it becomes

$$y+6+y=12;$$

from which we find that $y=3$; and since we have already seen that $x=y+6$, we find that $x=3+6=9$.

To verify these values, substitute them for x and y in the original equations, and we shall obtain

$$9+3=12,$$
$$9-3=\ \ 6.$$

Ex. 2. Again: take the equations

$$\left.\begin{aligned} 2x+3y&=13 \\ 5x+4y&=22. \end{aligned}\right\}$$

From the first equation we find

$$y=\frac{13-2x}{3}.$$

Substituting this value of y in the second equation, it becomes

$$5x+4\times\frac{13-2x}{3}=22,$$

an equation containing only x, which, when solved, gives

$$x=2;$$

and this value of x, substituted in either of the original equations, gives

$$y=3.$$

(109.) The method thus exemplified is expressed in the following

Rule.

Find an expression for the value of one of the unknown quantities in either of the equations; then substitute this value in the place of its equal in the other equation.

Ex. 3. Find the values of x and y in the equations

$$3x+y=13 \text{ and } 2x-y=2.$$

Ans. $x=3$, $y=4$.

Ex. 4. Find the values of x and y in the equations

$$3x-2y=11, \text{ and } 2x+5y=39.$$

Ans. $x=7$, $y=5$.

Ex. 5. Find the values of x and y in the equations

$$2x+7y=65, \text{ and } 6x-2y=34.$$

Ans. $x=8$, $y=7$.

Ex. 6. Find the values of x and y in the equations

$$4x+6y=84, \text{ and } 8x-3y=18.$$

Ans. $x=6$, $y=10$.

Ex. 7. Find the values of x and y in the equations

$$5x-9y=14, \text{ and } 8x-6y=56.$$

Ans. $x=10$, $y=4$.

Ex. 8. Find the values of x and y in the equations

$$7x-15y=75, \text{ and } 10x+3y=156.$$

Ans. $x=15$, $y=2$.

Ex. 9. Find the values of x and y in the equations

$$8x+7y=208, \text{ and } 5x+6y=156.$$

Ans. $x=12$, $y=16$.

Ex. 10. Find the values of x and y in the equations

$$10x+8y=202, \text{ and } 9x-4y=25.$$

Ans. $x=9$, $y=14$.

Ex. 11. A market-woman sold to one boy 4 apples

Quest.—Give the rule for elimination by substitution.

and 3 pears for 17 cents, and to another 7 apples and 5 pears for 29 cents. She sold them in each case at the same rate. What was the price of each?

Ans. The apples 2, and the pears 3 cents each.

Ex. 12. There are two brothers, A and B. If twice A's age be added to three times B's age, the sum will be 54; but if twice B's age be subtracted from five times A's age, the remainder will be 40. What are their ages?

Ans. A is 12 years and B 10 years.

Ex. 13. Two farmers, A and B, driving their sheep to market, says A to B, if you will give me half of your sheep, I shall have 75. Says B to A, if you will give me one third of your sheep, I shall have 75. How many had each?

Ans. A had 45 and B had 60.

Ex. 14. A grocer had two kinds of wine, marked A and B. He mixed together 3 gallons of A and 2 gallons of B, and sold the mixture at $1.35 per gallon. He also mixed 4 gallons of A and 6 of B, and sold the mixture at $1.40 per gallon. What was the value of each kind of wine?

Ans. A was worth $1.25 per gallon.
B was worth $1.50 per gallon.

Ex. 15. Find two numbers such that their sum shall be 42, and the greater shall be equal to five times the ess. *Ans.* 7 and 35.

(110.) ELIMINATION BY COMPARISON.

Ex. 1. To illustrate this method, take the equations

$$x+y=12,$$
$$x-y=6.$$

Derive from each equation an expression for x in terms of y, and we shall have

$$x=12-y,$$
$$x=\ 6+y.$$

These two values of x must be equal to each other, and, by *comparing* them, we shall obtain

$$12-y=6+y,$$

an equation involving only one unknown quantity; whence we obtain

$$y=3.$$

Substituting this value of y in the expression $x=6+y$, and we find $x=9$.

Ex. 2. Again: take the equations

$$2x+3y=13,$$
$$5x+4y=22.$$

From equation first we find

$$y=\frac{13-2x}{3},$$

and from equation second,

$$y=\frac{22-5x}{4}.$$

Putting these values of y equal to each other, we have

$$\frac{13-2x}{3}=\frac{22-5x}{4},$$

an equation containing only x; whence, by clearing of fractions and transposing, we obtain

$$x=2.$$

Substituting this value of x in either of the preceding expressions for y, we find

$$y=3.$$

(111.) The method thus exemplified is expressed in the following

Rule.

Find an expression for the value of the same unknown quantity in each of the equations, and form a new equation by putting one of these values equal to the other.

Ex. 3. Find the values of x and y in the equations

$$2x+3y=22, \text{ and } \frac{x}{2}-\frac{y}{2}=3.$$

Ans. $x=8$, $y=2$.

Ex. 4. Find the values of x and y in the equations

$$3x+y=14, \text{ and } 5x-y=10.$$

Ans. $x=3$, $y=5$.

Ex. 5. Find the values of x and y in the equations

$$5x+2y=43, \text{ and } x-\frac{y}{2}=5.$$

Ans. $x=7$, $y=4$.

Ex. 6. Find the values of x and y in the equations

$$4x+5y=51, \text{ and } \frac{3x}{2}+y=13.$$

Ans. $x=4$, $y=7$.

Ex. 7. Find the values of x and y in the equations

$$4x-2y=12, \text{ and } \frac{x}{2}-\frac{y}{5}=2.$$

Ans. $x=8$, $y=10$.

Ex. 8. Find the values of x and y in the equations

$$6x-9y=33, \text{ and } 5x-y=47.$$

Ans. $x=10$, $y=3$.

Quest.—Give the rule for elimination by comparison.

Ex. 9. Find the values of x and y in the equations

$$8x-3y=36, \text{ and } \frac{x}{3}+\frac{y}{4}=6.$$

Ans. $x=9$, $y=12$.

Ex. 10. Find the values of x and y in the equations

$$7x-4y=73, \text{ and } \frac{x}{5}+\frac{y}{2}=7.$$

Ans. $x=15$, $y=8$.

Ex. 11. A boy bought 3 apples and 5 oranges for 26 cents; he afterward bought, at the same rate, 4 apples and 7 oranges for 36 cents. How much were the apples and oranges apiece?

Ans. The apples were 2 cents and the oranges 4 cents.

Ex. 12. A market-woman sells to one person 6 quinces and 4 melons for 72 cents; and to another, 4 quinces and 2 melons, at the same rate, for 40 cents. How much are the quinces and melons apiece?

Ans. The quinces are 4 cents and the melons 12 cents apiece.

Ex. 13. In the market I find I can buy 4 bushels of barley and 3 bushels of oats for \$2; and, at the same price, 8 bushels of barley and 1 bushel of oats for \$3. What is the price of each per bushel?

Ans. The barley is 35 cents and the oats 20 cents.

Ex. 14. Six yards of broadcloth and ten yards of taffeta cost \$56; and, at the same rate, eight yards of broadcloth and 12 yards of taffeta cost \$72. What is the price of a yard of each?

Ans. The broadcloth cost \$6 and the taffeta \$2.

Ex. 15. A person expends one dollar in apples and pears, buying his apples at 2 for a cent and his pears at 2 cents a piece; afterward he accommodates his neighbor with $\frac{1}{2}$ of his apples and $\frac{1}{3}$ of his pears for 38 cents. How many of each did he buy?

Ans. 56 apples and 36 pears.

(112.) ELIMINATION BY ADDITION AND SUBTRACTION.

Ex. 1. To illustrate this method, take the two equations

$$x+y=12,$$
$$x-y=6.$$

Since the coefficients of y in the two equations are equal, and have contrary signs, we may eliminate this letter by adding the two equations together, whence we obtain

$$2x=18,$$

or

$$x=9.$$

We may now deduce the value of y by substituting the value of x in one of the original equations. Substituting in the first equation, we have

$$9+y=12,$$

whence

$$y=3.$$

Since the coefficients of x are equal in the two original equations, we might have eliminated this letter by subtracting one equation from the other. Subtracting the second from the first, we obtain

$$2y=6,$$

or

$$y=3.$$

Ex. 2. Again: let us take the equations

$$2x+3y=13,$$
$$5x+4y=22.$$

We perceive that if we could deduce from the proposed equations two other equations in which the co efficients of y should be equal, the elimination of y might be effected by ***subtracting*** one of these new equations from the other.

It is easily seen that we shall obtain two equations of the form required if we multiply all the terms of each equation by the coefficient of y in the other. Multiplying, therefore, all the terms of equation first by 4, and all the terms of equation second by 3, they become

$$8x+12y=52,$$
$$15x+12y=66.$$

Subtracting the former of these equations from the latter, we find

$$7x=14,$$

whence $\quad x=2.$

In like manner, in order to eliminate x, multiply the first of the proposed equations by 5, and the second by 2; they will become

$$10x+15y=65,$$
$$10x+8y=44.$$

Subtracting the latter of these two equations from the former, we have

$$7y=21,$$

whence $\quad y=3.$

(113.) This last method is expressed in the following

RULE.

1. ***Multiply or divide the equations, if necessary, in such a manner that one of the unknown quantities shall have the same coefficient in both.***

QUEST.—Give the rule for elimination by addition and subtraction.

7

2. *If the signs of the coefficients are the same in both equations, subtract one equation from the other; but, if the signs are unlike, add them together.*

Ex. 3. Find the values of x and y in the equations

$$3x+4y=59, \text{ and } 5x+7y=102.$$

Ans. $x=5$, and $y=11$.

Ex. 4. Find the values of x and y in the equations

$$\frac{x}{3}+\frac{y}{5}=10, \text{ and } \frac{x}{2}+\frac{y}{6}=11.$$

Ans. $x=12$, and $y=30$.

Ex. 5. Find the values of x and y in the equations

$$\frac{x}{5}+y=16, \text{ and } x+\frac{y}{7}=12.$$

Ans. $x=10$, $y=14$.

Ex. 6. Find the values of x and y in the equations

$$\frac{x}{8}+\frac{y}{4}=7, \text{ and } 3x-\frac{y}{5}=44.$$

Ans. $x=16$, $y=20$.

Ex. 7. Find the values of x and y in the equations

$$\frac{x}{2}-\frac{y}{5}=5, \text{ and } \frac{x}{7}+\frac{y}{2}=7.$$

Ans. $x=14$, $y=10$.

Ex. 8. Find the values of x and y in the equations

$$x+\frac{y}{2}=21, \text{ and } \frac{x}{8}+5y=52.$$

Ans. $x=16$, $y=10$.

Ex. 9. Find the values of x and y in the equations

$$\frac{x}{6}+\frac{y}{10}=4, \text{ and } \frac{x}{4}+\frac{y}{2}=13.$$

Ans. $x=12$, $y=20$.

Ex. 10. Find the values of x and y in the equations

$$\frac{x}{5}+3y=56, \text{ and } \frac{x}{2}-\frac{y}{6}=2.$$

Ans. $x=10$, $y=18$.

Ex. 11. My shoemaker sends me a bill of $9 for 1 pair of boots and 2 pair of shoes. Some months afterward he sends me a bill of $16 for 2 pair of boots and 3 pair of shoes. What are the boots and shoes a pair?

Ans. The boots are $5 and the shoes $2.

Ex. 12. A gentleman employs 5 men and 4 boys to labor one day, and pays them $7; the next day he hires, at the same wages, 8 men and 6 boys, and pays them $11. What are the daily wages of each?

Ans. The men have one dollar and the boys 50 cents.

Ex. 13. A vintner sold at one time 15 bottles of port and 10 bottles of sherry, and for the whole received $30. At another time he sold 12 bottles of port and 18 bottles of sherry, at the same prices as before, and for the whole received $39. What was the price of a bottle of each sort of wine?

Ans. The port was $1 and the sherry $1.50.

Ex. 14. A gentleman has two horses and one chaise. The first horse is worth $120. If the first horse be harnessed to the chaise, they will together be worth twice as much as the second horse. But if the second be harnessed, the horse and chaise will be worth three times the value of the first. What is the value of the second horse and of the chaise?

Ans. The second horse is worth $160 and the chaise $200.

Ex. 15. A farmer bought a quantity of rye and wheat for \$18, the rye at 75 cents and the wheat at \$1 per bushel. He afterward sold $\frac{1}{3}$ of his rye and $\frac{1}{4}$ of his wheat, at the same rate, for \$5. How many bushels were there of each?

Ans. 8 bushels of rye and 12 bushels of wheat.

(114.) The same example may be solved by either of the preceding methods, and each method has its advantages in particular cases. Generally, however, the first two methods give rise to fractional expressions which occasion inconvenience in practice, while the third method is not liable to this objection. When the coefficient of one of the unknown quantities in one of the equations is equal to unity, this inconvenience does not occur, and the method by substitution may be preferable; the third will, however, commonly be found most convenient. The following examples may be solved by either of these methods.

Ex. 1. Given $5x+4y=68$, and $3x+7y=73$, to find the values of x and y.

Ans. $x=8$, and $y=7$.

Ex. 2. Given $11x+3y=100$, and $4x-7y=4$, to find the values of x and y.

Ans. $x=8$, $y=4$.

Ex. 3. Given $\frac{x}{2}+\frac{y}{3}=7$, and $\frac{x}{3}+\frac{y}{2}=8$, to find the values of x and y. *Ans.* $x=6$, $y=12$.

Ex. 4. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$, to find the values of x and y.

Ans. $x=19$, $y=3$.

QUEST.—May the same example be solved by either method? Which method is to be preferred?

Ex. 5. Given $\frac{x}{7}+\frac{2y}{3}=26$, and $\frac{x}{5}-\frac{y}{4}=8$, to find the values of x and y.

Ans. $x=70$, $y=24$.

Ex. 6. It is required to find two numbers such that if half the first be added to the second, the sum will be 34, and if one third of the second be added to the first, the sum will be 28.

Ans. The numbers are 20 and 24.

Ex. 7. It is required to find two numbers such that their sum shall be 49, and the greater shall be equal to six times the less.

Ans. 7 and 42.

Ex. 8. The sum of two numbers is 33, and if 5 times the less be taken from 3 times the greater, the remainder will be 35. What are the numbers?

Ans. 8 and 25.

Ex. 9. The mast of a ship consists of two parts; one sixth of the lower part, added to one half of the upper part, is equal to 35 feet; and 3 times the lower part, diminished by 6 times the upper part, is equal to 30 feet. What is the height of the mast?

Ans. 130 feet.

Ex. 10. Two persons, A and B, talking of their money, says A to B, give me one third of your money, and I shall have 110 dollars. Says B to A, give me one fourth of your money, and I shall have 110 dollars. How much had each?

Ans. A had 80 dollars and B 90 dollars.

SECTION IX.

EQUATIONS OF THE FIRST DEGREE CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(115.) IN the preceding examples of two unknown quantities, the conditions of each problem have furnished two equations independent of each other. In like manner, if a problem involve three or more unknown quantities, it must furnish as many independent equations as there are unknown quantities.

Ex. 1. Take the system of equations

$$x+y+z=9, \quad (1.)$$
$$2x+3y+z=17, \quad (2.)$$
$$6x+5y+z=31. \quad (3.)$$

In order to eliminate z, let us subtract equation (1) from equation (2); we thus obtain

$$x+2y=8. \quad (4.)$$

Also, subtracting equation (2) from equation (3), we obtain

$$4x+2y=14. \quad (5.)$$

We have now obtained two equations (4) and (5) containing but two unknown quantities, and we may proceed as in Section VIII. Subtracting equation (4) from equation (5), we find

QUEST.—When a problem involves three or more unknown quantities, how many equations must it furnish?

$$3x=6,$$

or $$x=2.$$

Substituting this value of x in equation (4), we obtain

$$y=3.$$

Substituting these values of x and y in equation (1), we obtain

$$2+3+z=9,$$

whence $$z=4.$$

These values of x, y, and z may be verified by substitution in the original equations.

(116.) We have effected the elimination in this case by method third, Art. 113; but either of the other methods might have been employed. Hence, to solve three equations containing three unknown quantities, we have the following

Rule.

Eliminate one of the unknown quantities so as to obtain two equations containing only two unknown quantities; then from these two equations deduce another containing only one unknown quantity.

Ex. 2. Given $\left.\begin{array}{r} x+\ y+z=11, \\ 4x+2y+z=21, \\ 10x+5y+z=45, \end{array}\right\}$ to find x, y, and z.

Ans. $x=2$, $y=4$, $z=5$.

Ex. 3. Given $\left.\begin{array}{r} x+3y+5z=\ 56, \\ 2x+4y+7z=\ 79, \\ 3x+6y+9z=108, \end{array}\right\}$ to find x, y, and z.

Ans. $x=3$, $y=6$, $z=7$.

Quest.—Give the rule for solving three equations with three unknown quantities.

Ex. 4. Given $\left.\begin{aligned} 2x+3y+5z&=61, \\ x+4y+7z&=66, \\ 3x+6y+8z&=104, \end{aligned}\right\}$ to find x, y, and z.

Ans. $x=10,\ y=7,\ z=4.$

Ex. 5. Given $\left.\begin{aligned} 2x+6y+5z&=93, \\ 4x+3y+8z&=95, \\ 5x+4y+9z&=116, \end{aligned}\right\}$ to find x, y, and z.

Ans. $x=7,\ y=9,\ z=5.$

Ex. 6. Given $\left.\begin{aligned} x+y+z&=29, \quad (1.) \\ x+2y+3z&=62, \quad (2.) \\ \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z&=10, \quad (3.) \end{aligned}\right\}$ to find x, y, and z.

Subtracting equation (1) from (2), we obtain

$$y+2z=33. \qquad (4.)$$

Clearing equation (3) of fractions, we have

$$6x+4y+3z=120. \qquad (5.)$$

Multiplying equation (1) by 6, we obtain

$$6x+6y+6z=174. \qquad (6.)$$

Subtracting (5) from (6), we have

$$2y+3z=54. \qquad (7.)$$

We have thus obtained two equations, (4) and (7), containing only two unknown quantities.

Multiplying equation (4) by 2, we have

$$2y+4z=66. \qquad (8.)$$

Subtracting (7) from (8), we have

$$z=12.$$

Substituting this value of z in equation (7), we obtain

$$2y+36=54,$$

whence $y=9.$

Substituting these values of y and z in equation (1), we find $x+9+12=29$

whence $x=8.$

These values may be verified as in former examples.

Ex. 7. Given $\left.\begin{array}{r}2x+4y-3z=22,\\4x-2y+5z=18,\\6x+7y-\ z=63,\end{array}\right\}$ to find x, y, and z.

Ans. $x=3$, $y=7$, $z=4$.

Ex. 8. Given $\left.\begin{array}{r}x+\ y+2z=34,\\2x+\ y-\ z=45,\\x-2y+\ z=\ 7,\end{array}\right\}$ to find x, y, and z.

Ans. $x=20$, $y=8$, $z=3$.

Ex. 9. Given $\left.\begin{array}{r}x+2y-3z=13,\\3x+\ y+4z=51,\\\frac{1}{3}x+\frac{1}{4}y+\frac{1}{2}z=\ 7,\end{array}\right\}$ to find x, y, and z.

Ans. $x=9$, $y=8$, $z=4$.

Ex. 10. Given $\left.\begin{array}{r}x+\frac{1}{2}y+\frac{1}{3}z=32,\\\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15,\\\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12,\end{array}\right\}$ to find x, y, and z.

Ans. $x=12$, $y=20$, $z=30$.

Ex. 11. A market-woman sold to one boy 5 apples, 9 pears, and 10 peaches for 53 cents; and to another, 12 apples, 4 pears, and 6 peaches for 38 cents; and to a third, 8 apples, 11 pears, and 12 peaches for 66 cents. She sold them each time at the same rate. What was the price of each?

Ans. An apple, 1 cent;
a pear, 2 cents;
a peach, 3 cents.

Ex. 12. A market-woman sold at one time 12 eggs, 10 apples, and a pie for 35 cents; at another time, 10 eggs, 16 apples, and 2 pies for 40 cents; and at a third time, 18 eggs, 12 apples, and 4 pies for 66 cents. She sold these articles each time at the same rate. What was the price of each?

Ans. An egg, 2 cents;
an apple, $\frac{1}{2}$ cent;
a pie, 6 cents.

Ex. 13. A grocer had three kinds of wine, marked A, B, and C. He mixed together 10 gallons of A, 4 gallons of B, and 2 gallons of C, and sold the mixture at $1.30 per gallon. He also mixed together 8 gallons of A, 4 of B, and 4 of C, and sold the mixture at $1.35 per gallon. At another time he mixed 3 gallons of A, 2 gallons of B, and 11 gallons of C, and sold the mixture at $1.50 per gallon. What was the value of each kind of wine?

Ans. A was worth $1.20 per gallon;
B was worth 1.40 "
C was worth 1.60 "

Ex. 14. The sum of the distances which three persons, A, B, and C, have traveled, is 300 miles. A's distance added to C's is equal to twice B's; and three times A's added to twice B's, is equal to seven times C's. What are their respective distances?

Ans. A's is 120 miles;
B's is 100 miles;
C's is 80 miles.

Ex. 15. Three persons, A, B, and C, talking of their money, says A to B, give me half of your money, and I shall have $100; says B to C, give me one third of your money, and I shall have $100; says C to A, give me one fourth of your money, and I shall have $100. How much money had each?

Ans. A had $64, B had $72, and C had $84.

(117.) If we had *four* equations containing *four* unknown quantities, we might, by the methods already explained, eliminate one of the unknown quantities. We should thus obtain *three* equations between *three* unknown quantities, which might be solved according to Art. 116. We might proceed in a similar manner if we had *five* equations, or even a greater number.

Either of the unknown quantities may be selected as the one to be first exterminated. It is, however, generally best to begin with that which has the smallest coefficients; and if each of the unknown quantities is not contained in all the proposed equations, it is generally best to begin with that which is found in the least number of equations.

Ex. 16. Given $\left.\begin{array}{r} x+y+z+u=14, \\ 2x+y+z-u=6, \\ 2x+3y-z+u=14, \\ x-y+3z+4u=31, \end{array}\right\}$ to find x, y, z, and u.

Ans. $x=2$, $y=3$, $z=4$, $u=5$.

Ex. 17. Given $\left.\begin{array}{r} x+y+2z=22, \\ 2x+3u=33, \\ 3y+4z=43, \\ 4y+5u=65, \end{array}\right\}$ to find x, y, z, and u.

Ans. $x=3$, $y=5$, $z=7$, $u=9$.

Ex. 18. A market-woman sold at one time 6 eggs, 10 apples, and a pie for 19 cents; at another time, 10 eggs, 24 apples, and 2 pies for 37 cents; at a third time, 12 eggs, 18 apples, and 14 pears for 41 cents; and at a fourth time, 12 apples, 6 pies, and 18 pears for 54 cents. She sold each article constantly at the

QUEST.—How do we proceed with four unknown quantities? Which of the unknown quantities should be first exterminated?

same price as at first. What was the price of each article?

Ans. An egg, $1\frac{1}{2}$ cents;
an apple, $\frac{1}{2}$ cent.
a pie, 5 cents;
a pear, 1 cent.

Ex. 19. A grocer had four kinds of wine, marked A, B, C, and D. He mixed together 2 gallons of A, 14 gallons of B, and 4 gallons of C, and sold the mixture at $1.42 per gallon. He also mixed together 10 gallons of A, 5 gallons of C, and 5 gallons of D, and sold the mixture at $1.45 per gallon. At another time he mixed 10 gallons of B, 4 gallons of C, and 2 gallons of D, and sold the mixture at $1.50 per gallon. At another time he mixed together 6 gallons of A and 12 of D, and sold the mixture at $1.60 per gallon. What was the value of each kind of wine?

Ans. A was worth $1.20 per gallon:
B was worth 1.40 "
C was worth 1.60 "
D was worth 1.80 "

Ex. 20. It is required to find four numbers such that the sum of the first three shall be 30; the sum of the first, second, and fourth shall be 33; the sum of the first, third, and fourth shall be 35; and the sum of the last three shall be 37.

Ans. 8, 10, 12, and 15.

SECTION X.

INVOLUTION AND POWERS.

(118.) INVOLUTION is the process of raising a quantity to any proposed power.

The products formed by the successive multiplication of the same number by itself are called the powers of that number. See Art. 26. Thus

The first power of 3 is 3.

The square, or second power of 3, is 9, or 3×3.

The cube, or third power of 3, is 27, or $3\times3\times3$.

The fourth power of 3 is 81, or $3\times3\times3\times3$, etc., etc., etc.

(119.) *The exponent is a number or letter written a little above a quantity to the right, and denotes the number of times that quantity enters as a factor into a product.* See Art. 27.

Thus the first power of a is a^1, where the exponent is 1, which, however, is commonly written a, the exponent being understood.

The second power of a is $a\times a$, or a^2, where the exponent 2 denotes that a is taken twice as a factor to produce the power aa.

The third power of a is $a\times a\times a$, or a^3, where the

---

QUEST.—Give the definition of the word power. Define the term exponent.

exponent 3 denotes that a is taken three times as a factor to produce the power aaa.

The fourth power of a is $a \times a \times a \times a$, or a^4, etc.

PROBLEM I.

(120.) *To square a monomial.*

Let it be required to raise the monomial $3ab^2$ to the second power.

According to the rule for the multiplication of monomials, Arts. 55 and 56,

$$(3ab^2)^2 = 3ab^2 \times 3ab^2 = 9a^2b^4.$$

So, also, $(4a^2bc^3)^2 = 4a^2bc^3 \times 4a^2bc^3 = 16a^4b^2c^6$.

Hence, in order to square a monomial, we have the following

RULE.

Square the coefficient and multiply the exponent of each of the letters by 2.

Examples.

1. What is the square of $7axy$? *Ans.* $49a^2x^2y^2$.
2. What is the square of $11a^3bcd^2$? *Ans.* $121a^6b^2c^2d^4$.
3. What is the square of $8a^2b^3x^4$? *Ans.* $64a^4b^6x^8$.
4. What is the square of $12a^2xy$? *Ans.*
5. What is the square of $15ab^2cx^4$? *Ans.*
6. What is the square of $18x^2yz^3$? *Ans.*

(121.) We have seen in Art. 62 that + multiplied

QUEST.—Explain the method of squaring a monomial. Give the rule for squaring a monomial.

by +, and − multiplied by −, give +. Now the square of any quantity being the product of that quantity by itself, it follows that

Whatever may be the sign of a monomial, its square will be positive.

7. What is the square of $-10ab^2x^4$?

   Ans. $+100a^2b^4x^8$

8. What is the square of $-9a^2b^2c^3$?

   Ans. $+81a^4b^4c^6$.

9. What is the square of $16mx^2y^3$?

   Ans.

10. What is the square of $-20a^2bx^2y^3$?

    Ans.

Problem II.

(122.) *To raise a monomial to any power.*

Let it be required to raise the monomial $2a^2b^3$ to the fifth power.

According to the rules for multiplication,

$$(2a^2b^3)^5=2a^2b^3\times 2a^2b^3\times 2a^2b^3\times 2a^2b^3\times 2a^2b^3$$
$$=32a^{10}b^{15};$$

where we perceive,

1st. That the coefficient 2 has been raised to the fifth power.

2d. That the exponent of each of the letters has been multiplied by 5.

In like manner,

$$(3a^2b^3c)^3=3a^2b^3c\times 3a^2b^3c\times 3a^2b^3c$$
$$=27a^6b^9c^3;$$

where we perceive that the coefficient 3 has been rais-

Quest.—What will be the sign of the square? How do we raise a monomial to any power?

ed to the third power, and that the exponent of each of the letters has been multiplied by 3.

Hence, to raise a monomial to any power, we have the following

RULE.

1. *Raise the numerical coefficient to the given power by actual multiplication.*

2. *Multiply the exponent of each of the letters by the exponent of the power required.*

Examples.

1. What is the cube of $6xy^2z^4$?

   Ans. $216x^3y^6z^{12}$.

2. What is the fourth power of $4ab^2c^3$?

   Ans. $256a^4b^8c^{12}$?

3. What is the fifth power of $3ax^3y^5$?

   Ans. $243a^5x^{15}y^{25}$.

4. What is the fourth power of $3a^2bcx^3$?

   Ans.

5. What is the cube of $5a^2mx^3y^4$? *Ans.*

6. What is the cube of $7ab^3m^2y^3$? *Ans.*

(123.) We have seen that, whatever may be the sign of a monomial, its square is always positive. It is obvious that the product of several factors which are all *positive* is invariably positive. Also, the product of an *even* number of negative factors is *positive*, but the product of an *odd* number of negative factors is *negative*. Thus

---

QUEST.—Give the rule for raising a monomial to any power? How do we determine the sign of the power?

$$-a\times-a=+a^2;$$
$$-a\times-a\times-a=-a^3;$$
$$-a\times-a\times-a\times-a=+a^4;$$
$$-a\times-a\times-a\times-a\times-a=-a^5,$$

etc., etc., etc.

Hence we perceive that

Every* EVEN *power is positive, but an* ODD *power has the same sign as its root.

7. What is the cube of $-2ac^3x^5$?

Ans. $-8a^3c^9x^{15}$.

8. What is the fourth power of $-7ab^2d^3y^5$?

Ans. $+2401a^4b^8d^{12}y^{20}$.

9. What is the fifth power of $2c^3mx^2$?

Ans. $+32c^{15}m^5x^{10}$.

10. What is the fifth power of $3bc^2dy^2$?

Ans. $+243b^5c^{10}d^5y^{10}$.

11. What is the fourth power of $-4ad^6x^5$?

Ans. $+256a^4d^{24}x^{20}$.

12. What is the cube of $-8acx^2y^3$?

Ans. $-512a^3c^3x^6y^9$.

13. What is the cube of $-4ab^2my^2$?

Ans. $-64a^3b^6m^3y^6$.

14. What is the fifth power of $-2bc^2d^2x^3$?

Ans.

15. What is the fourth power of $5c^2x^3y^5$?

Ans.

16. What is the cube of $9ab^2d^2m^4y^5$? *Ans.*

17. What is the cube of $11a^2d^2x^3$? *Ans.*

18. What is the fourth power of $-6abc^2m^3y^3$?

Ans.

QUEST.—What is the rule for the sign of the power?

19. What is the fourth power of $-10cd^4x^4y^2$? *Ans.*

20. What is the fifth power of $-3a^2b^2x^3$? *Ans.*

PROBLEM III.

(124.) *To raise a fraction to any power.*

A fraction is raised to a given power by multiplying the fraction by itself; that is, by multiplying the numerator by itself and the denominator by itself, Art. 85.

Ex. 1. Thus the square of $\frac{a}{b}$ is

$$\frac{a}{b}\times\frac{a}{b}, \text{ which equals } \frac{a^2}{b^2}.$$

2. So, also, the cube of $\frac{a}{b}$ is

$$\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}, \text{ which equals } \frac{a^3}{b^3}.$$

3. What is the square of $\frac{5a^2b}{m}$? *Ans.* $\frac{25a^4b^2}{m^2}$.

4. What is the cube of $\frac{2ab^2}{3c}$? *Ans.* $\frac{8a^3b^6}{27c^3}$.

5. What is the fourth power of $\frac{3a^2x}{b^2}$?

Ans. $\frac{81a^8x^4}{b^8}$.

6. What is the fifth power of $\frac{2b^3x^2}{am}$?

Ans. $\frac{32b^{15}x^{10}}{a^5m^5}$.

QUEST.—How do we raise a fraction to any power?

7. What is the fourth power of $\frac{3ab^2x^3}{c^2}$?

Ans.

8. What is the cube of $\frac{8ab^2y^2}{mx}$? *Ans.*

9. What is the square of $\frac{7ax^2y^3}{bc}$? *Ans.*

10. What is the square of $\frac{6am^2x^4}{ny}$? *Ans.*

Problem IV.

(125.) *To raise a polynomial to any power.*

A polynomial is raised to a given power by multiplying the quantity continually by itself.

Ex. 1. Let it be required to find the fourth power of $a+b$.

$$\begin{array}{rl} & a+b \\ & a+b \\ \hline & a^2+ab \\ & \quad +ab+b^2 \\ \hline (a+b)^2= & a^2+2ab+b^2, \text{ the second power of } a+b. \\ & a+b \\ \hline & a^3+2a^2b+ab^2 \\ & \quad +a^2b+2ab^2+b^3 \\ \hline (a+b)^3= & a^3+3a^2b+3ab^2+b^3, \text{ the third power.} \\ & a+b \\ \hline & a^4+3a^3b+3a^2b^2+ab^3 \\ & \quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline (a+b)^4= & a^4+4a^3b+6a^2b^2+4ab^3+b^4, \text{ the fourth power.} \end{array}$$

QUEST.—How do we raise a polynomial to any power?

(126.) It will be observed that the number of multiplications is one less than the exponent of the power. Thus, to obtain the second power, we multiply the quantity by itself once; to obtain the third power, we multiply the quantity by itself twice, etc.

Exponents may be applied to polynomials as well as to monomials. Thus

$(a+b)^4$ denotes the fourth power of the expression $a+b$.

2. Find the fourth power of the binomial $a-b$.

$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad -ab+b^2} \end{array}$$

$(a-b)^2=a^2-2ab+b^2$, the second power of $a-b$.

$$\begin{array}{l} \underline{a-b} \\ a^3-2a^2b+ab^2 \\ \underline{\quad -a^2b+2ab^2-b^3} \end{array}$$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$, the third power.

$$\begin{array}{l} \underline{a-b} \\ a^4-3a^3b+3a^2b^2-ab^3 \\ \underline{\quad -a^3b+3a^2b^2-3ab^3+b^4} \end{array}$$

$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$, the fourth power.

3. What is the cube of $2a-1$?

Ans. $8a^3-12a^2+6a-1$.

4. What is the square of $a+b+c$?

Ans. $a^2+b^2+c^2+2ab+2ac+2bc$.

QUEST.—How does the number of multiplications compare with the exponent of the power?

5. What is the square of $2a^2-3b^2$?

Ans. $4a^4-12a^2b^2+9b^4$

6. What is the cube of $2a-3b$?

Ans. $8a^3-36a^2b+54ab^2-27b^3$.

7. What is the cube of $2ab+cd$?

Ans. $8a^3b^3+12a^2b^2cd+6abc^2d^2+c^3d^3$.

8. What is the fourth power of $3a-b$?

Ans. $81a^4-108a^3b+54a^2b^2-12ab^3+b^4$.

9. What is the cube of $a+b+c$?

Ans. $a^3+b^3+c^3+3ab^2+3ac^2+3a^2b+3a^2c+3bc^2+3b^2c+6abc$.

10. What is the cube of $2a-2ab+b^2$? *Ans.*

Binomial Theorem.

(127.) By the method already explained, any power of a binomial may be obtained by actual multiplication; but by the *binomial theorem* this labor may be greatly abridged.

The successive powers of the binomial $a+b$, found by actual multiplication, are as follows:

$$(a+b)^1=a+b$$
$$(a+b)^2=a^2+2ab+b^2$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$
$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

The powers of $a-b$, found in the same manner, are as follows:

QUEST.—What is the object of the binomial theorem?

$$(a-b)^1=a-b$$
$$(a-b)^2=a^2-2ab+b^2$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$
$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$
$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$$
$$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

(128.) On comparing the powers of $a+b$ with those of $a-b$, we perceive that they only differ in the *signs of certain terms*. In the powers of $a+b$, all the terms containing the *odd* powers of b have the sign $-$, while the *even* powers retain the sign $+$. The reason of this is obvious; for, since $-b$ is the only negative term of the root, the terms of the power can only be rendered negative by b. A term which contains the factor $-b$ an even number of times, will, therefore, be positive; a term which contains $-b$ an odd number of times, must be negative. Hence we conclude,

1*st*. *When both terms of the binomial are positive, all the terms of the power are positive.*

2*d*. *When the second term of the binomial is negative, all the odd terms of each power, counted from the left, are positive, and all the even terms negative.*

(129.) Also, we perceive that the *number of terms* in the power is always greater by unity than the exponent of the power. Thus the second power of the binomial contains three terms, the third power contains four terms, the fourth power contains five terms, etc.

QUEST.—How do the powers of $a+b$ compare with those of $a-b$? What terms will be positive and what will be negative? What is the number of terms in the power?

(130.) *Of the Exponents.*

If we consider the exponents of the preceding powers, we shall find that they follow a very simple law. Thus:

In the square, the exponents	of a are 2, 1, 0, of b are 0, 1, 2.
In the cube, the exponents	of a are 3, 2, 1, 0, of b are 0, 1, 2, 3.
In the fourth power, the exponents	of a are 4, 3, 2, 1, 0, of b are 0, 1, 2, 3, 4,

etc., etc., etc.

In the first term of each power, a is raised to the required power of the binomial; and in the following terms the exponent of a diminishes by unity until we reach the last term which does not contain a.

The exponent of b in the second term is 1, and increases by unity in each term to the right, until we reach the last term in which the exponent is the same as that of the required power.

(131.) The sum of the exponents of a and b in any term is equal to the exponent of the given power. Thus, in the second power, the sum of the exponents of a and b in each term is 2; in the third power it is 3; in the fourth power it is 4, etc. This remark will enable us to detect any error so far as regards the exponents.

We hence infer that, for the seventh power, the terms without the coefficients must be

$$a^7,\ a^6b,\ a^5b^2,\ a^4b^3,\ a^3b^4,\ a^2b^5,\ ab^6,\ b^7;$$

QUEST.—What are the exponents of a and b in the different powers? What law do the exponents follow? What is the sum of the exponents of a and b in any term?

and for the eighth power,

$$a^8,\ a^7b,\ a^6b^2,\ a^5b^3,\ a^4b^4,\ a^3b^5,\ a^2b^6,\ ab^7,\ b^8.$$

(132.) *Of the Coefficients.*

The coefficient of the first term in every power of $a+b$ is unity. The coefficient of the second term is the same as the exponent of the given power; and if *the coefficient of any term be multiplied by the exponent of* a *in that term, and divided by the exponent of* b *increased by one, it will give the coefficient of the succeeding term.*

Thus the fifth power of $a+b$ is

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

The coefficient of the second term is 5, which is the exponent of the power. Then, to find the coefficient of the third term, we multiply 5 by 4, the exponent of a in the second term, and divide by 2, which is the exponent of b increased by one. The quotient 10 is the coefficient of the third term. So, also, if 10, the coefficient of the third term, be multiplied by 3, the exponent of a, and divided by 3, the exponent of b increased by one, we obtain 10, the coefficient of the fourth term. Again: if 10, the coefficient of the fourth term, be multiplied by 2, the exponent of a, and divided by 4, the exponent of b increased by one, we obtain 5, the coefficient of the fifth term.

The coefficients of the sixth power will also be found as follows:

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6};$$

that is, 1, 6, 15, 20, 15, 6, 1.

QUEST.—What law do the coefficients follow?

The coefficients of the seventh power will be

$$1,\ 7,\ \frac{7\times6}{2},\ \frac{21\times5}{3},\ \frac{35\times4}{4},\ \frac{35\times3}{5},\ \frac{21\times2}{6},\ \frac{7\times1}{7};$$

that is,

$$1,\ 7,\ 21,\quad 35,\quad 35,\quad 21,\quad 7,\quad 1.$$

Therefore the seventh power of $a+b$ is

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

(133.) The following therefore, is the

Binomial Theorem.

In any power of a binomial a+b, *the exponent of* a *begins in the first term with the exponent of the power, and in the following terms continually decreases by one. The exponent of* b *commences with one in the second term of the power, and continually increases by one.*

The coefficient of the first term is one; that of the second is the exponent of the power; and if the coefficient of any term be multiplied by the exponent of a *in that term, and divided by the exponent of* b *increased by one, it will give the coefficient of the succeeding term.*

If we examine the powers of $a+b$ in Art. 127, we shall find that after we pass the middle term, the same coefficients are repeated in the inverse order. Thus the coefficients of

$(a+b)^5$ are 1, 5, 10, 10, 5, 1;
of $(a+b)^6$ are 1, 6, 15, 20, 15, 6, 1.

Hence it is only necessary to compute the coefficients for *half* the terms; we then repeat the same numbers in the inverse order.

Quest.—Repeat the binomial theorem. Is it necessary to compute all the coefficients according to the rule?

Examples.

Ex. 1. Raise $a+b$ to the 9th power.

The terms without the coefficients are

$$a^9,\ a^8b,\ a^7b^2,\ a^6b^3,\ a^5b^4,\ a^4b^5,\ a^3b^6,\ a^2b^7,\ ab^8,\ b^9.$$

And the coefficients are

$$1,\ 9,\ \frac{9\times 8}{2},\ \frac{36\times 7}{3},\ \frac{84\times 6}{4},\ \frac{126\times 5}{5},\ \frac{126\times 4}{6},\ \frac{84\times 3}{7},\ \frac{36\times 2}{8},\ \frac{9\times 1}{9};$$

that is,

1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

Prefixing the coefficients, we obtain

$$(a+b)^9=a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9.$$

It should be remembered that, according to a former remark, it is only necessary to compute the coefficients of *half* the terms independently.

Ex. 2. What is the seventh power of $x+y$?

Ans.

Ex. 3. What is the sixth power of $x-a$?

Ans.

Ex. 4. What is the fifth power of $m+n$?

Ans.

(134.) If the terms of the given binomial are affected with coefficients or exponents, they must be raised to the required powers according to the principles already established for the involution of monomials.

Ex. 5. What is the fourth power of $a-3b$?

Ans.

QUEST.—When the terms of the binomial have coefficients or exponents, how do we proceed?

For convenience, let us substitute x for $3b$; then

$$(a-x)^4=a^4-4a^3x+6a^2x^2-4ax^3+x^4.$$

But $x^2=9b^2$; $x^3=27b^3$; and $x^4=81b^4$.

Substituting for x its value, we have

$(a-3b)^4=a^4-12a^3b+54a^2b^2-108ab^3+81b^4$, *Ans.*

Ex. 6. What is the cube of $2a-3b$?

Ans. $8a^3-36a^2b+54ab^2-27b^3$.

Ex. 7. What is the square of $5a-7b$?

Ans. $25a^2-70ab+49b^2$.

Ex. 8. What is the cube of $4x-5y$?

Ans. $64x^3-240x^2y+300xy^2-125y^3$.

Ex. 9. What is the fifth power of $x-3y$?

Ans. $x^5-15x^4y+90x^3y^2-270x^2y^3+405xy^4-243y^5$.

Ex. 10. What is the fourth power of $3a^2-2b$?

Ans. $81a^8-216a^6b+216a^4b^2-96a^2b^3+16b^4$.

Ex. 11. What is the fourth power of $2x+5a^2$?

Ans. $16x^4+160x^3a^2+600x^2a^4+1000xa^6+625a^8$.

Ex. 12. What is the fourth power of $2x+4y$?

Ans. $16x^4+128x^3y+384x^2y^2+512xy^3+256y^4$

Ex. 13. What is the fourth power of $a+b+c$?

Substitute x for $b+c$; then

$$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

Restoring the value of x, we have

$$(a+b+c)^4=a^4+4a^3(b+c)+6a^2(b+c)^2+4a(b+c)^3 +(b+c)^4;$$

or, expanding the powers of $b+c$, we obtain

$$(a+b+c)^4=a^4+4a^3b+4a^3c+6a^2b^2+12a^2bc+6a^2c^2+4a +12ab^2c+12abc^2+4ac^3+b^4+4b^3c+6b^2c^2+4bc^3+$$

Ex. 14. What is the fourth power of $a+b-2c$?

Ans. $a^4+4a^3b-8a^3c+6a^2b^2-24a^2bc+24a^2c^2+4$
$-24ab^2c+48abc^2-32ac^3+b^4-8b^3c+24b^2c^2-3$
$+16c^4$.

SECTION XI.

EVOLUTION AND RADICAL QUANTITIES.

(135.) EVOLUTION is the process of finding the root of any quantity.

The *square*, or second power of a number, is the product arising from multiplying that number by itself once. Thus the square of 8 is 8×8, or 64; the square of 15 is 15×15, or 225.

The *square root* of a number is that number which, multiplied by itself once, will produce the given number.

Thus the square root of 144 is 12, because 12 multiplied by itself produces 144.

PROBLEM I.

(136.) *To Extract the Square Root of Numbers.*

If a number is a perfect square and is not very large, its root may generally be found by inspection. Thus the first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Hence the numbers in the first line are the square roots of the corresponding numbers in the second.

QUEST.—What is evolution? What is the square root of a number? How may the root sometimes be found?

But if the number is large, the discovery of its root may be attended with some difficulty. The following principles will, however, assist us in detecting the root.

(137.) I. *For every two figures of the square there will be one figure in the root, and also one for any odd figure.*

Thus the square of		1 is	1;
"	"	9 is	81;
"	"	10 is	100;
"	"	99 is	9801;
"	"	100 is	10000;
"	"	999 is	998001;
"	"	1000 is	1000000;

etc., etc., etc.

Hence we see that the square root of every number composed of one or two figures will contain *one* figure; the square root of every number composed of three or four figures will contain *two* figures; the square root of a number composed of five or six figures will contain *three* figures, etc.

Hence, if we divide the number into periods of two figures, proceeding from right to left, the number of figures in the root will be equal to the number of periods.

(138.) II. *The first figure of the root will be the square root of the greatest square number contained in the first period on the left.*

Every number consisting of more than one figure may be regarded as composed of a certain number of

QUEST.—How may we know the number of figures in the root? What will be the first figure of the root? How may the square of a number of two figures be decomposed?

tens and a certain number of units. If we represent the tens by a and the units by b, the number may be represented by

$$a+b,$$

whose square is $a^2+2ab+b^2$.

Hence we see that the square of a number composed of tens and units contains *the square of the tens plus twice the product of the tens by the units, plus the square of the units.*

Now the square of tens can give no significant figure in the first right-hand period; the square of hundreds can give no figure in the first two periods on the right, and the square of the highest figure in the root can give no figure except in the first period on the left.

Ex. 1. Let it be required to extract the square root of 2916.

Since this number is composed of four figures, its root will contain two figures; that is, it will consist of a certain number of tens and a certain number of units. Now the square of the tens must be found in the two left-hand figures, 29, which form the first period, and which we will separate from the other two figures by placing a point between them. Now this period contains not only the square of the tens, but also a part of the product of the tens by the units. The greatest square contained in 29 is 25, whose root is 5; hence 5 must be the number of tens whose square is 2500; and if we subtract this from 2916, the remainder, 416, contains twice the product of the tens by the units, plus the square of the units. If then, we divide this number by twice the tens, we

QUEST.—Explain the method of extracting the square root.

shall obtain the units, or possibly a number somewhat too large. This quotient figure can never be too small, but it may be too large, because the remainder, 416, besides twice the product of the tens by the units, contains the square of the units. We therefore complete the divisor by annexing the quotient 4 to the right of the 10, and then, multiplying by 4, we evidently obtain the double product of the tens by the units, plus the square of the units. The entire operation may then be represented as follows:

	29·16	54 = the root.
	25	
5×2=104	416	
	416	

In this operation we have, for convenience, written 25 for 2500. The two ciphers are, however, to be regarded as implied. So, also, the divisor is properly 50×2, or 100, which is contained in 416 four times; but we usually omit the last cipher in the divisor, calling it 5×2, or 10, and also omit the last figure of the dividend, leaving 41, which gives the same result as if all the figures were retained, for 10 is contained in 41 four times.

(139.) Hence, for the extraction of the square root of numbers, we derive the following

Rule.

1. *Separate the given number into periods of two figures each, beginning at the right hand. The first period on the left will often contain but one figure.*

Quest.—Give the rule for extracting the square root of numbers.

2. *Find the greatest square contained in the left-hand period: its root will be the first figure of the required root. Subtract the square from the first period, and to the remainder bring down the second period for a dividend.*

3. *Double the root already found for a divisor, and find how many times it is contained in the dividend, exclusive of its right-hand figure; annex the result both to the root and the divisor.*

4. *Multiply the divisor, thus increased, by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend. If the product should be greater than the dividend, diminish the last figure of the root.*

5. *Double the whole root now found for a new divisor, and continue the operation as before, until all the periods are brought down.*

Ex. 2. Find the square root of 186624.

The operation is as follows:

```
     18·66·24|432
     16      |
 83 | 2 66
    | 2 49
 862|  17 24
    |  17 24
```

Consequently, the required root is 432.

Ex. 3. Find the square root of 8836.

Ans. 94

Ex. 4. Find the square root of 58564.

Ans. 242

Ex. 5. Find the square root of 214369.

Ans. 463.

Ex. 6. Find the square root of 393129.

Ans. 627.

Ex. 7. Find the square root of 758641.

Ans. 871.

(140.) If, after all the periods have been brought down, there is no remainder, the proposed number is a perfect square. But if there is a remainder, we have only found the *entire part* of the root. In this case ciphers may be annexed forming new periods, each of which will give one decimal place in the root.

Ex. 8. Find the square root of 2972.

The operation is as follows:

```
        29·72 | 54.516+
        25    |
104   |  472
      |  416
1085  |  5600
      |  5425
10901 |  17500
      |  10901
109026|  659900
      |  654156
         5744 Rem.
```

Consequently, the square root of 2972 is 54.516, with a remainder of 5744. By annexing a greater number of ciphers, the root may be obtained to a greater number of decimal places; but, however far the operation may be carried, we shall always find a remainder.

QUEST.—What must be done when there is a remainder after all the periods have been brought down?

Ex. 9. Find the square root of 929296.

Ans. 964.

Ex. 10. Find the square root of 18957316.

Ans. 4354.

Ex. 11. Find the square root of 47348161.

Ans. 6881.

Ex. 12. Find the square root of 77158656.

Ans. 8784.

Ex. 13. Find the square root of 88078225.

Ans. 9385.

Ex. 14. Find the square root of 87.

Ans. 9.3273+.

Ex. 15. Find the square root of 158.

Ans. 12.5698+.

Ex. 16. Find the square root of 523.

Ans. 22.8691+.

Ex. 17. Find the square root of 654.

Ans. 25.5734+.

Ex. 18. Find the square root of 763.

Ans. 27.6224+.

Ex. 19. Find the square root of 2.

Ans. 1.4142+.

Ex. 20. Find the square root of 3.

Ans. 1.7320+.

PROBLEM II.

To Extract the Square Root of Fractions.

(141.) The second power of a fraction is obtained by multiplying the numerator into itself, and the denominator into itself. Thus the second power of $\frac{a}{b}$ is

---

QUEST.—How do we obtain the square root of a fraction?

$$\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

Hence *the square root of a fraction is equal to the square root of the numerator divided by the square root of the denominator.*

Ex. 1. What is the square root of $\frac{4}{9}$? *Ans.* $\frac{2}{3}$.

Ex. 2. What is the square root of $\frac{9}{25}$?

Ans. $\frac{3}{5}$.

Ex. 3. What is the square root of $\frac{36}{49}$?

Ans. $\frac{6}{7}$.

Ex. 4. What is the square root of $\frac{225}{729}$?

Ans. $\frac{15}{27}$.

Ex. 5. What is the square root of $\frac{1225}{5041}$?

Ans. $\frac{35}{71}$.

Ex. 6. What is the square root of $\frac{6889}{8281}$?

Ans. $\frac{83}{91}$.

(142.) If either the numerator or denominator is not a perfect square, we may *change the vulgar fraction into a decimal, continuing the division until the*

QUEST.—When the terms of the fraction are not perfect squares, how must we proceed?

number of decimal places is double the number of places required in the root. Then extract the root of the decimal fraction by Art. 140.

Ex. 7. What is the square root of $\frac{6}{17}$?

The fraction $\frac{6}{17}$, reduced to a decimal fraction, is .35294117; the square root of which is .59408 *Ans.*

Ex. 8. What is the square root of $\frac{3}{7}$?

Ans. .65465.

Ex. 9. What is the square root of $\frac{8}{13}$?

Ans. .78446.

(143.) The square root of a mixed quantity may be found in the usual way, if we reduce the vulgar fraction to its equivalent decimal, and divide the number into periods commencing with the decimal point.

Ex. 10. What is the square root of $3\frac{1}{3}$?

Ans. 1.8257.

Ex. 11. What is the square root of $10\frac{1}{4}$?

Ans. 3.2015.

Ex. 12. What is the square root of $7\frac{1}{5}$?

Ans. 2.6832.

Ex. 13. What is the square root of $29\frac{1}{7}$?

Ans. 5.3984.

Ex. 14. What is the square root of 32.462.

Ans. 5.6975.

Ex. 15. What is the square root of $75\frac{3}{7}$?

Ans. 8.6849.

QUEST.—How may we find the square root of a mixed quantity?

Ex. 16. What is the square root of $5\frac{7}{12}$?

Ans. 2.3629.

Ex. 17. What is the square root of $\frac{3}{5}$?

Ans. .77459.

Ex. 18. What is the square root of $\frac{2}{3}$?

Ans. .81649.

Ex. 19. What is the square root of 58.614336?

Ans. 7.656.

Ex. 20. What is the square root of 9.878449?

Ans. 3.143.

Problem III.

To Extract the Square Root of Monomials.

(144.) According to Art. 120, in order to square a monomial, we must square its coefficient and multiply the exponent of each of its letters by 2. Hence, in order to derive the square root of a monomial from its square, we have the following

Rule.

1. *Extract the square root of its coefficient.*
2. *Divide the exponent of each letter by 2.*

Ex. 1. Thus $\sqrt{64a^4b^2}=8a^2b$.

This is evidently the true result for

$$(8a^2b)^2=8a^2b\times 8a^2b=64a^4b^2.$$

Ex. 2. Find the square root of $81a^4b^6$.

Ans. $9a^2b^3$.

Ex. 3. Find the square root of $225a^8b^6c^2$.

Ans. $15a^4b^3c$.

Quest.—How do we extract the square root of a monomial?

Ex. 4. Find the square root of $361a^2b^6x^8$.

Ans. $19ab^3x^4$.

Ex. 5. Find the square root of $529a^2m^4x^6y^8$.

Ans. $23am^2x^3y^4$.

Ex. 6. Find the square root of $841a^6b^8c^4d^2$.

Ans. $29a^3b^4c^2d$.

(145.) It appears from the preceding rule that *a monomial can not be a perfect square unless its coefficient be a square number, and the exponent of each of its letters an even number.*

Thus $7ab^2$ is not a perfect square, for 7 is not a *square* number, and the exponent of a is not an *even* number. Its square root may be *indicated* by the usual sign thus:

$$\sqrt{7ab^2}.$$

Expressions of this kind are called *surds*, or *radicals of the second degree.*

A *radical quantity* is the indicated root of an imperfect power.

(146.) We have seen, Art. 121, that whatever may be the sign of a monomial, its square must be affected with the sign +. Hence we conclude that

If a monomial be positive, its square root may be either positive or negative.

Thus $\sqrt{9a^4}=+3a^2$, or $-3a^2$,

for either of these quantities, when multiplied by itself, produces $9a^4$. We therefore always affect the square root of a positive quantity with the double sign ±, which is read *plus* or *minus.*

QUEST.—What is necessary in order that a monomial may be a perfect square? What are surd quantities? What sign must we prefix to the square root of a monomial?

Thus $\sqrt{4a^6}=\pm 2a^3$.

$\sqrt{25a^2b^4}=\pm 5ab^2$.

(147.) If a monomial be *negative*, the extraction of its square root is *impossible*, since we have just seen that the square of every quantity, whether positive or negative, is necessarily positive.

Thus $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-5a}$,

are algebraic symbols indicating operations which it is impossible to execute. Quantities of this nature are called *imaginary* or *impossible* quantities, and are frequently met with in resolving equations of the second degree.

Problem IV.

To reduce Radicals to their most simple Forms.

(148.) Surds may frequently be simplified by the application of the following principle: *the square root of the product of two or more factors is equal to the product of the square roots of those factors:*

Or, in algebraic language,

$$\sqrt{ab}=\sqrt{a}\times\sqrt{b};$$

for each member of this equation, squared, will give the same quantity, viz., ab.

Let it be required to reduce $\sqrt{4a}$ to its most simple form.

This expression may be put under the form $\sqrt{4}\times\sqrt{a}$.

But $\sqrt{4}$ is equal to 2.

QUEST.—Can we extract the square root of a negative quantity? What are imaginary quantities? Upon what principle may radical quantities be simplified?

Hence $\sqrt{4a}=\sqrt{4}\times\sqrt{a}=2\sqrt{a}$.

$2\sqrt{a}$ is considered a *simpler* form than $\sqrt{4a}$.

Again: reduce $\sqrt{48}$ to its most simple form.

$\sqrt{48}$ is equal to $\sqrt{16\times3}=\sqrt{16}\times\sqrt{3}=4\sqrt{3}$.

(149.) Hence, in order to simplify a monomial radical of the second degree, we have the following

RULE.

Separate the expression into two factors, one of which is a perfect square; extract its root; and prefix it to the other factor, with the radical sign between them.

In the expressions $2\sqrt{a}$ and $4\sqrt{3}$, the quantities 2 and 4 are called the *coefficients of the radicals.*

To determine whether a given number contains a factor which is a perfect square, try whether it is divisible by either of the square numbers 4, 9, 16, 25, 36, etc.

Examples.

1. Reduce $\sqrt{18a^2}$ to its most simple form.

Ans. $3a\sqrt{2}$.

2. Reduce $\sqrt{75a^3b^4}$ to its most simple form.

Ans. $5ab^2\sqrt{3a}$.

3. Reduce $\sqrt{192a^2b}$ to its most simple form.

Ans. $8a\sqrt{3b}$.

4. Reduce $\sqrt{486a^5b^4c^2}$ to its most simple form.

Ans. $9a^2b^2c\sqrt{6a}$.

QUEST.—Give the rule for simplifying radical quantities. What are coefficients of radicals? How may we know whether a number contains a factor which is a square?

5. Reduce $\sqrt{432a^4b^3}$ to its most simple form.

Ans. $12a^2b\sqrt{3b}$.

6. Reduce $\sqrt{1125a^5bc^2}$ to its most simple form.

Ans. $15a^2c\sqrt{5ab}$.

7. Reduce $\sqrt{343a^3m^2x}$ to its most simple form.

Ans. $7am\sqrt{7ax}$.

8. Reduce $\sqrt{980a^6b^5c^2}$ to its most simple form.

Ans. $14a^3b^2c\sqrt{5b}$.

9. Reduce $\sqrt{2560a^9c^2x}$ to its most simple form.

Ans. $16a^4c\sqrt{10ax}$.

10. Reduce $\sqrt{1331a^7b^3x}$ to its most simple form.

Ans. $11a^3b\sqrt{11abx}$.

11. Reduce $\sqrt{864a^2b^5c^7}$ to its most simple form.

Ans. $12ab^2c^3\sqrt{6bc}$.

PROBLEM V.

To add Radical Quantities together.

(150.) Two radicals of the second degree are *similar*, when the quantities under the radical sign are the same in both.

Thus $3\sqrt{a}$ and $5\sqrt{a}$ are similar radicals.

So also are $2\sqrt{3}$ and $5\sqrt{3}$.

But $2\sqrt{3}$ and $3\sqrt{2}$ are *not* similar radicals.

Radicals may be added together by the following

RULE.

When the radicals are similar, add their coefficients, and to the sum annex the common radical.

QUEST.—What are similar radicals? Give the rule for adding radical quantities.

But if the radicals are not similar, and can not be made similar by the reductions in the preceding Article, they can only be connected together by the sign of addition.

Ex. 1. Find the sum of $2\sqrt{a}$ and $3\sqrt{a}$.

As these are similar radicals, we may unite their coefficients by the usual rule; for it is evident that twice the square root of a, and three times the square root of a, make five times the square root of a.

Ex. 2. Find the sum of $3\sqrt{6}$ and $5\sqrt{6}$.

Ans. $8\sqrt{6}$.

Ex. 3. Find the sum of $2m\sqrt{a}$ and $3n\sqrt{a}$.

Ans. $(2m+3n)\sqrt{a}$.

Ex. 4. Find the sum of $7\sqrt{3a}$ and $9\sqrt{3a}$.

Ans. $16\sqrt{3a}$.

Ex. 5. Find the sum of $m\sqrt{a+b}$ and $n\sqrt{a+b}$.

Ans. $(m+n)\sqrt{a+b}$.

(151.) If the radicals are originally dissimilar, they must, if possible, be made similar by the method of Art. 149.

Ex. 6. Find the sum of $\sqrt{27}$ and $\sqrt{48}$.

Here $\sqrt{27}=\sqrt{9\times3}=3\sqrt{3}$,

and $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$.

Whence their sum is $7\sqrt{3}$ *Ans.*

Ex. 7. Find the sum of $\sqrt{18}$ and $\sqrt{32}$.

Ans. $7\sqrt{2}$.

Ex. 8. Find the sum of $\sqrt{75}$ and $\sqrt{108}$.

Ans. $11\sqrt{3}$.

QUEST.—May radicals which appear to be dissimilar be sometimes united?

Ex. 9. Find the sum of $\sqrt{20a^2}$ and $\sqrt{45a^2}$.
Ans. $5a\sqrt{5}$.

Ex. 10. Find the sum of $\sqrt{54a^2}$ and $\sqrt{96a^2}$.
Ans. $7a\sqrt{6}$.

Ex. 11. Find the sum of $\sqrt{32am^2}$ and $\sqrt{50am^2}$.
Ans. $9m\sqrt{2a}$.

Ex. 12. Find the sum of $\sqrt{45a^2b^2x}$ and $\sqrt{125a^2b^2x}$.
Ans. $8ab\sqrt{5x}$.

Ex. 13. Find the sum of $2\sqrt{147}$ and $\sqrt{75}$.
Ans. $19\sqrt{3}$.

Ex. 14. Find the sum of $\sqrt{72}$ and $\sqrt{128}$.
Ans. $14\sqrt{2}$.

Ex. 15. Find the sum of $\sqrt{180}$ and $\sqrt{405}$.
Ans. $15\sqrt{5}$.

PROBLEM VI.

To find the Difference of Radical Quantities.

(152.) It is evident that the subtraction of radical quantities may be performed in the same manner as addition, except that the signs in the subtrahend are to be changed according to Art. 48. Hence we have the following

RULE.

When the radicals are similar, subtract their coefficients, and to the difference annex the common radical.

But if the radicals are not similar, and can not be made similar, their difference can only be indicated by the sign of subtraction.

QUEST.—Give the rule for the subtraction of radical quantities.

Ex. 1. Find the difference between $5m\sqrt{a}$ and $2m\sqrt{a}$.

Ans. $3m\sqrt{a}$.

Ex. 2. Find the difference between $7ab\sqrt{2}$ and $3ab\sqrt{2}$.

Ans. $4ab\sqrt{2}$.

Ex. 3. Find the difference between $\sqrt{75}$ and $\sqrt{27}$

Ans. $2\sqrt{3}$.

Ex. 4. Find the difference between $\sqrt{150}$ and $\sqrt{24}$.

Ans. $3\sqrt{6}$.

Ex. 5. Find the difference between $\sqrt{448}$ and $\sqrt{112}$.

Ans. $4\sqrt{7}$.

Ex. 6. Find the difference between $5\sqrt{20}$ and $3\sqrt{45}$.

Ans. $\sqrt{5}$.

Ex. 7. Find the difference between $2\sqrt{50}$ and $\sqrt{18}$.

Ans. $7\sqrt{2}$.

Ex. 8 Find the difference between $\sqrt{80a^4x}$ and $\sqrt{20a^4x}$.

Ans. $2a^2\sqrt{5x}$.

Ex. 9. Find the difference between $2\sqrt{72a^2}$ and $\sqrt{162a^2}$.

Ans. $3a\sqrt{2}$.

Ex. 10. Find the difference between $\sqrt{490am^2}$ and $\sqrt{40am^2}$.

Ans. $5m\sqrt{10a}$.

Problem VII.

(153.) *To multiply Radical Quantities together.*

Let it be required to multiply $\sqrt{a}$ by $\sqrt{b}$.

The product, $\sqrt{a}\times\sqrt{b}$, will be $\sqrt{ab}$.

For if we raise each of these quantities to the second power we obtain the same result, ab; hence these two expressions are equal. We therefore have the following

Rule.

Multiply the quantities under the radical signs together, and place the radical sign over the product.

If the radicals have coefficients, these must be multiplied together, and the product placed before the radical sign.

Ex. 1. What is the product of $3\sqrt{8}$ and $2\sqrt{6}$?

Ans. $6\sqrt{48}$ which equals $6\sqrt{16\times3}$, or $24\sqrt{3}$.

Ex. 2. What is the product of $5\sqrt{8}$ and $3\sqrt{5}$?

Ans. $15\sqrt{40}$ or $30\sqrt{10}$.

Ex. 3. What is the product of $2\sqrt{3}$ and $3\sqrt{5}$?

Ans. $6\sqrt{15}$.

Ex. 4. What is the product of $2\sqrt{18}$ and $3\sqrt{20}$?

Ans. $6\sqrt{360}$ or $36\sqrt{10}$.

Ex. 5. What is the product of $5\sqrt{2}$ and $3\sqrt{8}$?

Ans. $15\sqrt{16}$ or 6[illegible]

Ex. 6. What is the product of $2\sqrt{3ab}$ and $3\sqrt{2ab}$?

Ans. $6\sqrt{6a^2b^2}$ or $6ab\sqrt{6}$.

Ex. 7. What is the product of $7\sqrt{5}$ and $5\sqrt{15}$?

Ans. $35\sqrt{75}$ or $175\sqrt{3}$.

Ex. 8. What is the product of $2b\sqrt{xy}$ and $5b\sqrt{xy}$?

Ans. $10b^2xy$.

Ex. 9. What is the product of $2\sqrt{a^2b}$ and $5\sqrt{bc^2}$?

Ans. $10\sqrt{a^2b^2c^2}$ or $10abc$.

QUEST.—Give the rule for multiplying radical quantities

Ex. 10. What is the product of $2a\sqrt{15a}$ and $3a\sqrt{3a}$?

Ans. $6a^2\sqrt{45a^2}$ or $18a^2\sqrt{5}$.

Problem VIII.

(154.) *To divide one Radical Quantity by another.*

Let it be required to divide $\sqrt{a^3}$ by $\sqrt{a^2}$.

The quotient must be a quantity which, multiplied by the divisor, shall produce the dividend. We thus obtain $\sqrt{a}$; for, according to Art. 153,

$$\sqrt{a^2}\times\sqrt{a}=\sqrt{a^3};$$

that is,

$$\frac{\sqrt{a^3}}{\sqrt{a^2}}=\sqrt{a}.$$

Hence we have the following

Rule.

Divide one of the quantities under the radical sign by the other, and place the radical sign over the quotient.

If the radicals have coefficients, divide the coefficient of the dividend by the coefficient of the divisor, and place the quotient before the radical sign.

Ex. 1. Divide $8\sqrt{108}$ by $2\sqrt{6}$.

Ans. $4\sqrt{18}$ or $4\sqrt{9\times2}$, or $12\sqrt{2}$.

Ex. 2. Divide $4\sqrt{6a^2y}$ by $2\sqrt{3y}$.

Ans. $2\sqrt{2a^2}$ or $2a\sqrt{2}$.

Ex. 3. Divide $\sqrt{10a^2}$ by $\sqrt{5}$.

Ans. $\sqrt{2a^2}$ or $a\sqrt{2}$.

Ex. 4. Divide $4ab\sqrt{21}$ by $\sqrt{7}$.

Ans. $4ab\sqrt{3}$.

Quest.—Give the rule for the division of radical quantities

Ex. 5. Divide $15ab\sqrt{6xy}$ by $5b\sqrt{2y}$.

Ans. $3a\sqrt{3x}$.

Ex. 6. Divide $\sqrt{20x^3}$ by $\sqrt{5x}$.

Ans. $\sqrt{4x^2}$ or $2x$.

Ex. 7. Divide $6a\sqrt{48x^4}$ by $3\sqrt{4x^2}$.

Ans. $2a\sqrt{12x^2}$ or $4ax\sqrt{3}$.

Ex. 8. Divide $24\ ab^2\sqrt{12ax}$ by $12ab\sqrt{3a}$.

Ans. $2b\sqrt{4x}$ or $4b\sqrt{x}$.

Ex. 9. Divide $6a^2\sqrt{50x^5}$ by $3a\sqrt{5x}$.

Ans. $2a\sqrt{10x^4}$ or $2ax^2\sqrt{10}$

Ex. 10. Divide $14a^2b\sqrt{72a^2b^2}$ by $7a\sqrt{8ab}$.

Ans. $2ab\sqrt{9ab}$ or $6ab\sqrt{ab}$

Ex. 11. Divide $6a^2b^2c^2\sqrt{28}$ by $2a\sqrt{7}$.

Ans. $3ab^2c^2\sqrt{4}$ or $6ab^2c^2$

Ex. 12. Divide $30a^2b^3\sqrt{27a}$ by $15ab\sqrt{3a}$.

Ans. $2ab^2\sqrt{9}$ or $6ab^2$.

PROBLEM IX.

To Extract the Square Root of a Polynomial.

(155.). In order to discover a method for extracting the square root of a polynomial, let us consider the square of $a+b$, which is $a^2+2ab+b^2$. If we write the terms of the square in such a manner that the powers of one of the letters, as a, may go on continually decreasing, the first term will be the square of the first term of the root; and since, in the present case, the first term of the square is a^2, the first term of the root must be a.

QUEST.— Explain the method of extracting the square root of a polynomial.

Having found the first term of the root, we must consider the rest of the square, namely, $2ab+b^2$, to see how we can derive from it the second term of the root. Now this remainder, $2ab+b^2$, may be put under the form $(2a+b)b$; whence it appears that we shall find the second term of the root if we divide the remainder by $2a+b$. The first part of this divisor, $2a$, is double of the first term already determined; the second part, b, is yet unknown, and it is necessary at present to leave its place empty. Nevertheless, we may commence the division, employing only the term $2a$; but as soon as the quotient is found, which in the present case is b, we must put it in the vacant place, and thus render the divisor complete.

The whole process, therefore, may be represented as follows:

$$\begin{array}{l|l} a^2+2ab+b^2 & \underline{a+b}=\text{the root,} \\ \underline{a^2\qquad\qquad\quad} & \\ \quad 2ab+b^2 & 2a+b=\text{the divisor.} \\ \quad 2ab+b^2 & \end{array}$$

(156.) Hence, to extract the square root of a polynomial, we have the following

RULE.

1. *Arrange the terms in the order of the powers of some one letter; take the square root of the first term for the first term of the required root, and subtract its square from the given polynomial.*

2. *Divide the first term of the remainder by double the root already found, and annex the result both to the root and the divisor.*

QUEST.—Give the rule for extracting the square root of a polynomial.

3. *Multiply the divisor thus increased by the last term of the root, and subtract the product from the last remainder.*

4. *Double the terms of the root already found for a new divisor, and divide the first term of the remainder by the first term of the divisor, and annex the result both to the root and the divisor.*

5. *Multiply the divisor thus increased by the last term of the root, and subtract the product from the last remainder. Proceed in the same manner to find the additional terms of the root.*

Ex. 1. Required the square root of the polynomial

$$a^4+2a^3x+3a^2x^2+2ax^3+x^4.$$

The operation may be represented as follows:

$$
\begin{array}{l|l}
a^4+2a^3x+3a^2x^2+2ax^3+x^4 & a^2+ax+x^2=\text{the root,} \\
a^4 & \\
\hline
\quad 2a^3x+3a^2x^2 & 2a^2+ax=\text{the first divisor,} \\
\quad 2a^3x+a^2x^2 & \\
\hline
\qquad\quad 2a^2x^2+2ax^3+x^4 & 2a^2+2ax+x^2 = \text{the second divisor.} \\
\qquad\quad 2a^2x^2+2ax^3+x^4 & \\
\hline
\end{array}
$$

The terms of the polynomial being arranged in the order of the powers of the letter a, we extract the square root of a^4 and obtain a^2, which is the first term of the root. Having subtracted a^4 from the given polynomial, we divide $2a^3x$, the first term of the remainder, by $2a^2$, and obtain $+ax$, which is the second term of the root, which we annex both to the root and also to the divisor. We then multiply the complete divisor, $2a^2+ax$, by ax, and subtract the product from the last remainder. We now double the terms of the root a^2+ax, and obtain $2a^2+2ax$ for a new divisor.

Dividing $2a^2x^2$ by $2a^2$, we obtain x^2, the third term of the root, which we annex both to the root and to the divisor. Multiplying the complete divisor by x^2, and subtracting from the last remainder, nothing remains. Therefore a^2+ax+x^2 is the required root.

For verification, multiply the root by itself, and we shall obtain the original polynomial.

Ex. 2. Required the square root of the polynomial

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4.$$

Ans. a^2-ax+x^2.

Ex. 3. Required the square root of the polynomial

$$a^4-4a^3x+6a^2x^2-4ax^3+x^4.$$

Ans. $a^2-2ax+x^2$.

Ex. 4. Required the square root of the polynomial

$$a^4+4a^3x+4a^2x^2-12ax-6a^2+9.$$

Ans. $a^2+2ax-3$.

Ex. 5. Required the square root of the polynomial

$$a^4-4a^3b+8ab^3+4b^4.$$

Ans. $a^2-2ab-2b^2$.

Ex. 6. Required the square root of the polynomial

$$4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4.$$

Ans. $2x^2+2ax+4b^2$.

Ex. 7. Required the square root of the polynomial

$$9x^6-12x^5+10x^4-10x^3+5x^2-2x+1.$$

Ans. $3x^3-2x^2+x-1$.

Ex. 8. Required the square root of the polynomial

$$a^2+2ab+2ac+b^2+2bc+c^2.$$

Ans. $a+b+c$.

Ex. 9. Required the square root of the polynomial

$$4x^4-12x^3+13x^2-6x+1.$$

Ans. $2x^2-3x+1$.

QUEST.—How may the result obtained be verified?

Ex. 10. Required the square root of the polynomial
$$a^6+2a^5+3a^4+4a^3+3a^2+2a+1.$$
Ans. a^3+a^2+a+1.

(157.) *No binomial can be a perfect square.* For the square of a monomial is a monomial; and the square of a binomial consists of three distinct terms, which do not admit of being reduced to a less number. Thus the expression
$$a^2+b^2$$
is not a perfect square; it wants the term $\pm 2ab$ to render it the square of $a\pm b$. This remark should be continually borne in mind, as beginners often put the square root of a^2+b^2 equal to $a+b$.

(158.) *A trinomial is a perfect square when two of its terms are squares, and the third is the double product of the roots of these squares.* Thus

the square of $a+b$ is $a^2+2ab+b^2$,

and the square of $a-b$ is $a^2-2ab+b^2$.

Therefore the square root of $a^2\pm 2ab+b^2$ is $a\pm b$.

(159.) Hence, to obtain the square root of a trinomial when it is a perfect square, we have the following

RULE.

Extract the roots of the two terms which are complete squares, and connect them by the sign of the other term.

Ex. 11. Required the square root of $a^2+4ab+4b^2$.

The two terms a^2 and $4b^2$ are complete squares, and

QUEST.—Why can not a binomial be a perfect square? When is a trinomial a perfect square? Give the rule for the square root of a trinomial.

the third term, $4ab$, is twice the product of the roots a and $2b$; hence $a+2b$ is the root required.

Ex. 12. Required the square root of $9a^2-24ab+16b^2$
Ans. $3a-4b$.

Ex. 13. Required the square root of $9a^4-30a^3b+25a^2b^2$.
Ans.

Ex. 14. Required the square root of $4a^2+20ab+3b^2$.
Ans.

Ex. 15. Required the square root of $16a^6-26a^4b^2+49a^2b^4$.
Ans.

Ex. 16. Required the square root of $64a^6-49b^4$.
Ans.

There are other roots which may be obtained by successive extractions of the square root. Thus,

The *fourth* root is equal to the square root of the square root.

The *eighth* root is equal to the square root of the fourth root, &c.

Ex. 17. Required the fourth root of $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$.
Ans. $3a-2b$.

Ex. 18. Required the fourth root of $16a^4+b^4+x^4+24a^2b^2+24a^2x^2+6b^2x^2+32a^3b-32a^3x+8ab^3-8ax^3-4b^3x-4bx^3-48a^2bx-24ab^2x+24abx^2$.
Ans. $2a+b-x$.

Ex. 19. Required the fourth root of $16y^4+128y^3x+384y^2x^2+512yx^3+256x^4$.
Ans. $2y+4x$.

Ex. 20. Required the fourth root of $16a^4+160a^3b^2+600a^2b^4+1000ab^6+625b^8$.
Ans. $2a+5b^2$

SECTION XII.

EQUATIONS OF THE SECOND DEGREE.

(160.) *An equation of the second degree is one in which the highest power of the unknown quantity is a square.*

Thus, $4x^2+6x=28$,

and $ax^2+bx=c$,

are equations of the second degree.

(161.) Equations of the second degree are divided into two classes.

I. Equations which contain only the square of the unknown quantity and known terms. These are called *incomplete equations.*

Of this description are the equations

$$3x^2+12=150;\ ax^2=b, \text{ etc.}$$

II. Equations which contain both the first and second powers of the unknown quantity, together with known terms. These are called *complete equations.*

Of this description are the equations

$$x^2-10x=7;\ ax^2+bx=c, \text{ etc.}$$

INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

(162.) *Every incomplete equation of the second degree can be reduced to an equation containing but two terms.*

QUEST.—What is an equation of the second degree? What are incomplete equations of the second degree? What are complete equations? Every incomplete equation can be reduced to what form?

Thus, take the equation

$$4x^2-7=3x^2+9.$$

By transposition, we obtain

$$x^2=16.$$

Again: take the equation

$$\frac{x^2}{3}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{299}{24}.$$

Clearing the equation of fractions, we obtain

$$8x^2-72+10x^2=7-24x^2+299.$$

Transposing terms, we obtain

$$24x^2+8x^2+10x^2=7+299+72;$$

and uniting similar terms, we have

$$42x^2=378;$$

or, dividing each member by 42,

$$x^2=9.$$

Hence every incomplete equation of the second degree can be reduced to an equation of the form

$$x^2=a;$$

and, for this reason, incomplete equations are sometimes called *equations of two terms.*

(163.) If we extract the square root of each member of this equation, we obtain

$$x=\sqrt{a}.$$

Hence, to solve an incomplete equation of the second degree, we have the following

Rule.

Find the value of x^2, *and extract the square root of both members of the equation.*

QUEST.—Why are incomplete equations called equations of two terms? Give the rule for solving an incomplete equation.

Ex. 1. What is the value of x in the equation

$$5x^2-18=3x^2+14?$$

Transposing terms, we have

$$5x^2-3x^2=14+18.$$

Reducing, $2x^2=32$,

or $x^2=16$.

Extracting the square root,

$$x=4.$$

This value may be verified by substitution in the original equation. We thus obtain

$$5\times 4^2-18=3\times 4^2+14,$$

or $5\times16-18=3\times16+14$;

that is, $62=62$.

It should, however, be observed, that the square root of 16 is either $+4$ or -4, for $-4\times-4=16$; see Art. 62. And this value may also be verified by substitution in the original equation.

(164.) A *root* of an equation is the value of the unknown quantity in the equation. The preceding equation has two roots, viz., $+4$ and -4, and universally we shall find,

1*st. Every incomplete equation of the second degree has two roots.*

2*d. These roots are numerically equal, but have contrary signs.*

Ex. 2. What are the values of x in the equation

$$x^2-17=130-2x^2?$$

By transposition, $3x^2=147$.

QUEST.—What is a root of an equation? How many roots has an incomplete equation of the second degree? What relation have these roots to each other?

Therefore $$x^2=49,$$
and $$x=+7 \text{ or } -7.$$

Ex. 3. What are the values of x in the equation
$$7x^2-24=4x^2+51\,?$$
Ans. $x=+5$ or -5.

Ex. 4. What are the values of x in the equation
$$6x^2-48-2x^2=96\,?$$
Ans. $x=+6$ or -6.

Ex. 5. What are the values of x in the equation
$$x^2-36=\frac{x^2}{4}+12\,?$$
Ans. $x=+8$ or -8.

Ex. 6. What are the values of x in the equation
$$12+6x^2=\frac{2x^2}{3}+5x^2+15\,?$$
Ans. $x=+3$ or -3

Ex. 7. What are the values of x in the equation
$$\frac{4x^2+5}{9}=45\,?$$
Ans. $x=+10$ or -10.

Ex. 8. What are the values of x in the equation
$$x^2+ab=5x^2\,?$$
Ans. $x=+\frac{\sqrt{ab}}{2}$ or $-\frac{\sqrt{ab}}{2}$.

Ex. 9. What number is that which, being multiplied by itself, gives the product 256?

Ans. +16 or −16.

Ex. 10. What number is that the third part of whose square being subtracted from 18, leaves a remainder equal to 6?

Ans. +6 or −6.

Ex. 11. A boy, being asked his age, answered that if it were multiplied by itself, and from the product 108 were subtracted, the remainder would be the square of half his age. What was his age?

Ans. 12 years.

Ex. 12. What two numbers are those whose sum is to the greater as 10 to 7, and whose sum, multiplied by the less, produces 270?

Let $10x$ represent the sum.

Then $7x$ will represent the greater number, and $3x$ will represent the less.

Whence $30x^2=270$,

and $x^2=9$;

therefore $x=3$,

and the numbers are 21 and 9.

Ex. 13. What two numbers are those which are to each other as 4 to 5, and the difference of whose squares is 81?

Let $4x$ and $5x$ represent the numbers.

Ans. 12 and 15.

Ex. 14. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Let $9x$ and $7x$ represent the two numbers.

Ans. 18 and 14.

Ex. 15. What two numbers are those which are to each other as 2 to 3, and the sum of whose squares is 117?

Ans. 6 and 9.

Ex. 16. What two numbers are those whose difference is to the greater as 3 to 8, and the sum of whose squares is 356?

Ans. 10 and 16.

Ex. 17. Find two numbers which are to each other as 3 to 5, and whose product is 240.

Ans. 12 and 20.

Ex. 18. Find two numbers whose sum is to their difference as 5 to 2, and the difference of whose squares is 160?

Ans. 14 and 6.

Ex. 19. Find two numbers whose sum is to the less as 11 to 5, and whose sum multiplied by the less produces 220.

Ans. 12 and 10.

Ex. 20. A mercer bought a piece of silk for 324 shillings; and the number of shillings which he paid for a yard was to the number of yards as 4 to 9. How many yards did he buy, and what was the price of a yard?

Ans. 27 yards at 12 shillings per yard

COMPLETE EQUATIONS OF THE SECOND DEGREE.

(165.) A complete equation of the second degree is one which contains both the first and second powers of the unknown quantity, together with known terms.

Every complete equation of the second degree can be reduced to an equation containing but three terms.

Thus, take the equation

$$5x^2+18=3x^2-4x+48.$$

By transposing and reducing, we have

$$2x^2+4x=30;$$

and, dividing by 2, $x^2+2x=15$,

an equation containing but three terms.

QUEST.—What is a complete equation of the second degree? To what form may every complete equation be reduced?

Again take the equation

$$\frac{7x^2}{3}+12+x=\frac{4x^2}{3}+4x+16.$$

Clearing the equation of fractions, we obtain

$$7x^2+36+3x=4x^2+12x+48.$$

Transposing terms, we obtain

$$7x^2-4x^2+3x-12x=48-36.$$

Uniting similar terms, we have

$$3x^2-9x=12,$$

or, dividing by 3,

$$x^2-3x=4.$$

(166.) Hence every complete equation of the second degree can be reduced to an equation of the form

$$x^2+ax=b;$$

and, for this reason, complete equations are sometimes called *equations of three terms*. It is to be understood, however, that the signs of the quantities a and b may be either positive or negative.

Suppose we have the equation

$$x^2-6x+9=1,$$

in which it is required to find the value of x.

Since each member of this equation is a *complete square*, if we extract the square root, we shall obtain a new equation involving only the first power of x, which may be easily solved.

We thus have $x-3=\pm 1$,

and, by transposition,

$$x=3\pm 1=4 \text{ or } 2.$$

In order to verify these values, substitute each of

QUEST.—Why is a complete equation called an equation of three terms?

them in place of x in the given equation. Taking the first value, we shall have

$$4^2-6\times4+9=1;$$

that is, $$16-24+9=1,$$

an identical equation.

Taking the second value of x, we obtain

$$2^2-6\times2+9=1;$$

that is, $$4-12+9=1,$$

an identical equation.

Hence we see that a complete equation is readily solved, provided each member of the equation is a perfect square. But equations seldom occur under this form. Take, for example,

$$x^2-6x=-8.$$

The preceding method seems to be inapplicable, because the first member is not a complete square. We may, however, render it a complete square by the addition of 9, which must also be added to the second member to preserve the equality.

The equation thus becomes

$$x^2-6x+9=9-8=1,$$

which is the equation before proposed.

(167.) The peculiar difficulty, then, in resolving complete equations of the second degree, consists in rendering the first member an exact square.

In order to discover a general method of solution, let us take the equation

$$x^2+ax=b,$$

which is the form to which every complete equation of the second degree can be reduced.

Quest.—In what consists the difficulty of solving a complete equation? Explain the method of solving the general equation.

In order to avoid fractions, we will represent the coefficient of x by $2p$, and will write the equation in the form

$$x^2+2px=q.$$

We have seen that if we can by any transformation render the first member of this equation the perfect square of a binomial, we can reduce the equation to one of the first degree by extracting the square root.

Now we know that the square of a binomial, $x+p$, is $x^2+2px+p^2$; that is, the square is composed of the square of the first term, plus twice the product of the first term by the second, plus the square of the second term.

Hence, considering x^2+2px as the first two terms of the square of a binomial, we see that the third term of the square must be p^2, which is *the square of half the coefficient of the first power of* x.

If, then, we add p^2 to the first member of the proposed equation, it will become a complete square; and, in order that the equality may not be destroyed, we must add the same term to the second member of the equation, which thus becomes

$$x^2+2px+p^2=q+p^2.$$

Extracting the square root of each member of the equation, we have

$$x+p=\pm\sqrt{q+p^2};$$

whence, by transposing p, we have

$$x=-p\pm\sqrt{q+p^2}.$$

We prefix the double sign $\pm$, because the squares both of $+\sqrt{q+p^2}$ and also of $-\sqrt{q+p^2}$ is $+q+p^2$, and therefore every equation of the second degree must have two roots.

(168.) Hence, for the solution of every complete equation of the second degree, we have the following

RULE.

1. *Clear the equation of fractions. Transpose all the known quantities to one side of the equation, and all the terms containing the unknown quantity to the other side, and reduce the equation to the form* $x^2+2px=q$.

2. *Take half the coefficient of the first power of* x, *square it, and add the result to each member of the equation.*

3. *Extract the square root of both members of the equation, and transpose the known term contained in the first member to the other side of the equation.*

Ex. 1. What are the values of x in the equation

$$x^2-10x=-16?$$

Completing the square by adding to each member the square of half the coefficient of the second term, we have

$$x^2-10x+25=25-16=9.$$

Extracting the root,

$$x-5=\pm 3.$$

Hence $x=5\pm 3=8$ or 2.

Thus x has two values, either 8 or 2. To verify them, substitute in the original equation, and we shall have

$$8^2-10\times 8=-16, \quad 64-80=-16;$$

also, $2^2-10\times 2=-16$, *i. e.*, $4-20=-16$;

both of which are identical equations.

QUEST.—Give the rule for solving a complete equation of the second degree.

Ex. 2. What are the values of x in the equation

$$x^2-2x=24?$$

Completing the square,

$$x^2-2x+1=24+1=25.$$

Extracting the root, $x-1=\pm5$.

Whence $x=1\pm5=+6$ or -4.

These values of x may be verified as in the preceding example.

Ex. 3. What are the values of x in the equation

$$x^2+6x=-8?$$

Ans. $x=-2$ or -4.

Ex. 4. What are the values of x in the equation

$$x^2+6x=27?$$

Ans. $x=+3$ or -9.

Ex. 5. What are the values of x in the equation

$$2x^2+8x-20=70?$$

Ans. $x=+5$ or -9.

Ex. 6. What are the values of x in the equation

$$x^2+2x+4=\frac{x^2}{4}-x+28?$$

Ans. $x=+4$ or -8.

Ex. 7. What are the values of x in the equation

$$x^2-10x+50=\frac{x^2+14x-2}{5}?$$

Ans. $x=+9$ or $+7$.

Ex. 8. What are the values of x in the equation

$$3x^2-6x-74=31?$$

Ans. $x=+7$ or -5.

Ex. 9. What are the values of x in the equation

$$x^2+2x-\frac{10}{7}=90-\frac{x^2+2x}{7}?$$

Ans $x=+8$ or -10.

Ex. 10. What are the values of x in the equation

$$x^2+4x-15=\frac{45}{4}-x-\frac{x^2}{4}?$$

Ans. $x=+3$ or -7.

(169.) The preceding method of solving a complete equation is applicable to all cases; nevertheless, it sometimes leads to inconvenient fractions. For, in order to reduce a given equation to the required form, $x^2+2px=q$, we must divide by the coefficient of x^2, which it is often impossible to do without a remainder. Let, then, the equation be represented by the form

$$ax^2+bx=c.$$

Multiply each member of the equation by $4a$, and it becomes

$$4a^2x^2+4abx=4ac.$$

Adding b^2 to both members, we have

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

The first member of the equation is now a complete square, and its square root is $2ax+b$.

(170.) Hence, for completing the square, we may use the following

Rule.

Multiply the equation by four times the coefficient of x², *and add to both sides the square of the coefficient of* x.

If the coefficient of x^2 is unity, this rule becomes, *Multiply the equation by four, and add to each member the square of the coefficient of* x.

Either of these methods of completing the square

Quest.—What inconvenience sometimes results from the preceding method of solution? What method is free from this inconvenience?

may be practiced at pleasure; but the first method is to be preferred, except when its application would involve inconvenient fractions.

Ex. 11. What are the values of x in the equation

$$x^2-x-40=170.$$

Transposing, we obtain

$$x^2-x=210.$$

Multiplying by 4,

$$4x^2-4x=840.$$

Adding 1 to each member of the equation,

$$4x^2-4x+1=841.$$

Extracting the square root,

$$2x-1=\pm 29.$$

Whence, $2x=1\pm 29=+30$ or -28;

that is, $x=+15$ or -14.

Ex. 12 What are the values of x in the equation

$$3x^2+2x-9=76?$$

Transposing, $3x^2+2x=85$

Multiplying each member by 12, we have

$$36x^2+24x=1020.$$

Adding the square of 2 to each member, we obtain

$$36x^2+24x+4=1024.$$

Extracting the square root,

$$6x+2=\pm 32.$$

Whence $6x=-2\pm 32=+30$ or -34;

that is, $x=+5$ or $-5\frac{2}{3}$.

Ex. 13. What are the values of x in the equation

$$\frac{x^2}{2}-\frac{x}{3}+20\tfrac{1}{2}=42\tfrac{2}{3}?$$

Clearing of fractions and transposing, we have

QUEST.—When is the first method to be preferred?

$$3x^2-2x=133.$$

Completing the square and extracting the root,

$$6x=2\pm40=42 \text{ or } -38.$$

Whence $x=7$ or $-6\frac{1}{3}$.

Ex. 14. What are the values of x in the equation

$$\frac{x^2}{2}-\frac{x}{3}+7\frac{3}{8}=8\,?$$

Clearing of fractions, completing the square, and extracting the root, we have

$$24x=8\pm28=36 \text{ or } -20.$$

Whence $x=1\frac{1}{2}$ or $-\frac{5}{6}$.

Ex. 15. What are the values of x in the equation

$$4x^2-3x=85\,?$$

Ans. $x=5$ or $-\frac{17}{4}$.

Ex. 16. What are the values of x in the equation

$$6x+\frac{35-3x}{x}=44\,?$$

Ans. $x=7$ or $\frac{5}{6}$.

Ex. 17. What are the values of x in the equation

$$\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}\,?$$

Ans. $x=25$ or 1.

Ex. 18. What are the values of x in the equation

$$\frac{2x}{3}+\frac{1}{x}=\frac{7}{3}\,?$$

Ans. $x=3$ or $\frac{1}{2}$.

Ex. 19. What are the values of x in the equation

$$x+4=9+\frac{x^2-5x}{3}?$$

Ans. $x=3$ or 5.

Ex. 20. What are the values of x in the equation

$$2x+4=\frac{15}{2}+\frac{x^2-x}{4}?$$

Ans. $x=2$ or 7.

Ex. 21. What are the values of x in the equation

$$x^2+8x+50=35?$$

Ans. $x=-3$ or -5.

Ex. 22. What are the values of x in the equation

$$7x+35=11-3x-x^2?$$

Ans. $x=-4$ or -6.

Ex. 23. What are the values of x in the equation

$$\frac{3x^2}{4}+16x+17=5x-\frac{x^2}{4}+7?$$

Ans. $x=-1$ or -10.

Ex. 24. What are the values of x in the equation

$$\frac{2x^2}{7}+65+11x=17-\frac{5x^2}{7}-3x?$$

Ans. $x=-6$ or -8.

Ex. 25. What are the values of x in the equation

$$\frac{2x^2}{3}+4x+18=6-2x?$$

Ans. $x=-3$ or -6.

Ex. 26. What are the values of x in the equation

$$2x^2+9x+25=23+4x?$$

Ans. $x=-2$ or $-\frac{1}{2}$.

Ex. 27. What are the values of x in the equation

$$4x-\frac{36-x}{x}=46\,?$$

Ans. $x=12$ or $-\frac{3}{4}$.

Ex. 28. What are the values of x in the equation

$$4x-\frac{14-x}{x+1}=14\,?$$

Ans. $x=4$ or $-\frac{7}{4}$.

Ex. 29. What are the values of x in the equation

$$\frac{8x}{x+2}-6=\frac{20}{3x}\,?$$

Ans. $x=10$ or $-\frac{2}{3}$.

Ex. 30. What are the values of x in the equation

$$3x^2-2x=65\,?$$

Ans. $x=5$ or $-4\frac{1}{3}$.

(171.) PROBLEMS PRODUCING COMPLETE EQUATIONS OF THE SECOND DEGREE.

Prob. 1. It is required to find two numbers such that their difference shall be 6 and their product 160.

Let $x=$ the least number.

Then will $x+6=$ the greater.

And by the question, $x(x+6)=x^2+6x=160$.

Therefore, $x=10$ the less number,

$x+6=16$ the greater number.

Proof. $16-10=\ 6$, the first condition;

$16\times10=160$, the second condition.

Prob. 2. The receiving reservoir at Yorkville is a rectangle, 60 rods longer than it is broad, and its area is 5500 square rods. Required its length and breadth.

Ans. 50 and 110 rods.

Prob. 3. It is required to divide the number 50 into two such parts that their product shall be 544.

Ans. 34 and 16.

Prob. 4. In a bag which contains 110 coins of silver and gold, each gold coin is worth as many cents as there are silver coins, and each silver coin is worth as many cents as there are gold coins; and the whole are worth twenty dollars. How many are there of each?

Ans. 10 of gold and 100 of silver.

Prob. 5. There is a number consisting of two digits, whose sum is 10, and the sum of their squares is 58. Required the number.

Let $x=$ the first digit.

Then will $10-x=$ the second digit.

And $x^2+(10-x)^2=2x^2-20x+100=58$;

that is, $x^2-10x=-21$,

$$x^2-10x+25=4,$$

$$x=5\pm2=7 \text{ or } 3.$$

Hence the number is 73 or 37.

The two values of x are the required digits whose sum is 10. It will be observed that we put x to represent the first digit, whereas we find it may equal the second as well as the first. The reason is, that we have here imposed a condition which does not enter into the equation. If x represent *either* of the required digits, then $10-x$ will represent the *other*, and hence the values of x found by solving the equation should give both digits. Beginners are very apt thus, in the statement of a problem, to impose conditions which do not appear in the equation.

QUEST.—What error are beginners prone to commit?

The preceding example, and all others of the same class, may be solved without completing the square. Thus,

Let x represent the half difference of the two digits.

Then, according to the principle on page 122,

$5+x$ will represent the greater of the two digits,

$5-x$ will represent the less of the two digits.

The square of $5+x$ is $25+10x+x^2$.

The square of $5-x$ is $25-10x+x^2$.

The sum is $50+2x^2$,

which, according to the problem, equals 58.

Hence $2x^2=8$,

or $x^2=4$,

and $x=\pm 2$.

Therefore, $5+x=7$ the greater digit,

$5-x=3$ the less digit.

Prob. 6. A laborer dug two trenches, whose united length was 26 yards, for 356 shillings; and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

Ans. 10 or 16 yards.

Prob. 7. A farmer bought a number of sheep for 80 dollars, and if he had bought four more for the same money, he would have paid one dollar less for each. How many did he buy?

Let x represent the number of sheep.

Then will $\frac{80}{x}$ be the price of each.

And $\frac{80}{x+4}$ would be the price of each if he had bought four more for the same money.

But by the equation we have

$$\frac{80}{x}=\frac{80}{x+4}+1.$$

Solving this equation, we obtain

$x=16$ *Ans.*

Prob. 8. By selling my horse for 24 dollars, I lose as much per cent. as the horse cost me. What was the first cost of the horse?

Let x represent the first cost.

Then $\frac{x}{100}$ is the loss per cent.

And $\frac{x^2}{100}$ is the absolute loss.

Whence we find $x=40$ or 60 dollars.

Prob. 9. There are two numbers whose difference is 7, and half their product, plus 30, is equal to the square of the smaller number. What are the numbers?

Ans. 12 and 19.

Prob. 10. Divide the number 30 into two such parts that their product may be equal to eight times their difference.

Ans. 6 and 24.

Prob. 11. A and B set out at the same time to a place at the distance of 150 miles. A travels 4 miles an hour faster than B, and arrives at his journey's end 10 hours before him. At what rate did each travel per hour?

Ans. B 6 miles and A 10 miles per hour.

Prob. 12. Divide the number 33 into two such parts that their product shall be 162.

Ans. 27 and 6.

Prob. 13. What two numbers are those whose sum is 29 and product 100?

Ans. 25 and 4.

Prob. 14. The difference of two numbers is 6; and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

Ans. 1 and 7, or 11 and 17.

Prob. 15. There are two numbers whose sum is 30; and one third of their product, plus 18, is equal to the square of the less number. What are the numbers?

Ans. 9 and 21.

Prob. 16. A farmer bought a certain number of sheep for $120. If there had been 8 more, each sheep would have cost him half a dollar less? How many sheep were there?

Ans. 40.

Prob. 17. The plate of a looking-glass is 18 inches by 12, and is to be inclosed by a frame of uniform width throughout, whose area is to be equal to that of the glass. Required the width of the frame.

Ans. 3 inches.

Prob. 18. What two numbers are those whose sum is 19, and whose difference, multiplied by the greater, is 60?

Ans. 12 and 7.

Prob. 19. A boy, being asked his age, replied, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age.

Ans. 16.

Prob. 20. It is required to find a number whose square exceeds its first power by 210.

Ans. 15.

EQUATIONS OF THE SECOND DEGREE WITH MORE THAN ONE UNKNOWN QUANTITY.

(172.) An equation containing two unknown quantities, is said to be of the second degree when *the greatest sum of the exponents of the unknown quantities in any term is equal to two.* Thus,

$$3x^2-4x+y^2=25,$$

and
$$7xy-4x+y=40,$$

are equations of the second degree.

(173.) When we have given two such equations containing two unknown quantities, we may, by the methods of Art. 108–113, eliminate one of them, and obtain a new equation containing but one unknown quantity. The solution of two equations of the second degree containing two unknown quantities, generally involves the solution of an equation of the fourth degree containing one unknown quantity. Hence the principles hitherto established are not sufficient to enable us to solve *all* equations of this description. In some cases, however, the resulting equation is of the second degree, and may be solved by the preceding rules.

Ex. 1. Given $\left.\begin{array}{l} xy=81 \\ \frac{x}{y}=9 \end{array}\right\}$ to find the values of x and y.

QUEST.—What is an equation of the second degree containing two unknown quantities? How do we solve equations of the second degree containing two unknown quantities?

From the second equation, $x=9y$.

Substituting this value in the first equation, we have

$$9y^2=81.$$

Whence $y^2=9$,

and $y=\pm 3$.

Hence $x=\pm 27$.

Ex. 2. Given $\left.\begin{array}{r} x+y:x::5:3, \\ xy=6, \end{array}\right\}$ to find the values of x and y.

From the first equation, we find

$$3x+3y=5x.$$

Whence $y=\frac{2x}{3}$.

Substituting this value in the second equation, we obtain

$$\frac{2x^2}{3}=6.$$

Therefore, $x^2=9$,

and $x=\pm 3$.

Whence $y=\frac{2x}{3}=\pm 2$.

Ex. 3. Given $\left.\begin{array}{r} x^2+y^2=34, \\ xy=15, \end{array}\right\}$ to find the values of x and y.

Adding twice the second equation to the first, we obtain

$$x^2+2xy+y^2=64;$$

and extracting the square root,

$$x+y=8. \qquad \text{(a)}$$

Subtracting twice the second equation from the first, we obtain

$$x^2-2xy+y^2=4.$$

Whence $x-y=2. \qquad \text{(b)}$

Adding equation (a) to equation (b), we have

$$2x=10,$$

and $$x=5.$$

Whence, from equation (a), $y=3$.

Ex. 4. Given $\left.\begin{array}{r} x+y:x::7:5, \\ xy+y^2=126, \end{array}\right\}$ to find the values of x and y.

From the first equation we obtain

$$5x+5y=7x.$$

Whence $$x=\frac{5y}{2}.$$

Substituting this value for x in the second equation,

$$\frac{5y^2}{2}+y^2=126.$$

Whence $$y^2=36,$$

and $$y=\pm 6.$$

Therefore, $$x=\pm 15.$$

Ex. 5. Given $\left.\begin{array}{r} x+y:x-y::8:1, \\ xy=63, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 9$, $y=\pm 7$.

Ex. 6. Given $\left.\begin{array}{r} x+y=21, \\ x^2-y^2=63, \end{array}\right\}$ to find the values of x and y.

Ans. $x=12$, $y=9$.

Ex. 7. Given $\left.\begin{array}{r} x+y=23, \\ xy=120, \end{array}\right\}$ to find the values of x and y.

From the first equation we have

$$x=23-y.$$

Multiplying each member by y,

$$xy=23y-y^2=120.$$

Completing the square and extracting the root,

$$y-\frac{23}{2}=\pm\frac{7}{2}.$$

Whence $y=15$ or 8,

and $x=8$ or 15.

Ex. 8. Given $\left.\begin{aligned} x+y &= 20, \\ x^2+y^2 &= 218, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=13$.

Ex. 9. Given $\left.\begin{aligned} x-y &= 15, \\ x &= 2y^2, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=18$ or $\frac{25}{2}$, $y=3$ or $-\frac{5}{2}$.

Ex. 10. Given $\left.\begin{aligned} \frac{10x+y}{xy} &= 3, \\ 9y-9x &= 18, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=2$ or $-\frac{1}{3}$, $y=4$ or $\frac{5}{3}$.

Ex. 11. Given $\left.\begin{aligned} x+y:x-y::13:5&, \\ y^2+x=25&, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=9$, $y=4$.

Ex. 12. Given $\left.\begin{aligned} x^2+3xy-y^2 &= 23, \\ x+2y &= 7, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=2$.

Ex. 13. Given $\left.\begin{aligned} 2x^2+xy-5y^2 &= 20, \\ 2x-3y &= 1, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=3$.

Ex. 14. Given $\left.\begin{aligned} x^2+y^2 &= 281, \\ x^2-y^2 &= 231, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=16$, $y=5$.

Ex. 15. Given $\left.\begin{aligned} y^2+4x &= 2y+11, \\ x+4y &= 14, \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=2$ or -46, $y=3$ or 15.

Ex. 16. What two numbers are those whose difference, multiplied by the greater, produces 40; and whose difference, multiplied by the less, produces 15?

Let $x=$ the greater number,
and $y=$ the less number.

Then $x^2-xy=40$,
and $xy-y^2=15$.

Subtracting the second equation from the first,

$$x^2-2xy+y^2=25,$$

and extracting the square root,

$$x-y=\pm 5.$$

Therefore, from the first equation,

$$\pm 5x=40,$$

and $x=\pm 8$.

Also, $y=\pm 3$.

Ex. 17. What two numbers are those whose difference, multiplied by the less, produces 42; and whose difference, multiplied by their sum, produces 133?

Let $x=$ the greater number,
and $y=$ the less.

Then $xy-y^2=42$,
and $x^2-y^2=133$.

Subtracting twice the first equation from the second, we obtain

$$x^2-2xy+y^2=49.$$

Extracting the square root,

$$x-y=\pm 7.$$

Therefore, from the first equation,

$$\pm 7y=42,$$

and $y=\pm 6$.

Hence $x=\pm 13$.

Ex. 18. What number is that the sum of whose digits is 15; and if 31 be added to their product, the digits will be inverted?

Ans. 78.

Ex. 19. What two numbers are those whose product is 54, and quotient is 6?

Ans. 3 and 18.

Ex. 20. The product of two numbers is a, and their quotient is b. What are the numbers?

Ans. $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$.

Ex. 21. The sum of the squares of two numbers is 260, and the difference of their squares 132. What are the numbers?

Ans. 8 and 14.

Ex. 22. The sum of the squares of two numbers is a, and the difference of their squares is b. What are the numbers?

Ans. $\sqrt{\frac{a+b}{2}}$ and $\sqrt{\frac{a-b}{2}}$.

Ex. 23. Find two numbers which are to each other as 3 to 4, and the sum of whose squares is 400.

Ans. 12 and 16.

Ex. 24. Find two numbers which are to each other as 2 to 3, and the difference of whose squares is 125.

Ans. 10 and 15.

Ex. 25. Divide the number 16 into two parts, such that the product of the two parts, added to the sum of their squares, may be equal to 208.

Ans. 4 and 12.

Ex. 26. What two numbers are those whose product is 255, and the sum of whose squares is 514?

Ans. 15 and 17.

Ex. 27. What two numbers are those whose difference is 8, and the sum of whose squares is 544?

Ans. 12 and 20.

Ex. 28. What two numbers are those whose sum is 41, and the sum of whose squares is 901?

Ans. 15 and 26.

Ex. 29. What two numbers are those whose product is 120; and if the greater be increased by 8, and the less by 5, the product of the two numbers thus obtained shall be 300?

Ans. 12 and 10, or 16 and $7\frac{1}{2}$.

Ex. 30. Divide the number 100 into two such parts that the sum of their square roots may be 14.

Ans. 36 and 64.

Ex. 31. From two places at a distance of 720 miles, two persons, A and B, set out at the same time to meet each other. A traveled 12 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day. How many miles did each travel per day?

Ans. A 36 miles, and B 24 miles.

Ex. 32. A tailor bought a piece of cloth for $120, from which he cut four yards for his own use and sold the remainder for $120, gaining one dollar per yard. How many yards were there, and what did it cost him per yard? *Ans.* 24 yards at $5 per yard.

Ex. 33. The fore wheel of a carriage makes five revolutions more than the hind wheel in going 60 yards; but if the circumference of each wheel be increased one yard, it will make only three revolutions more than the hind wheel in the same space. Required the circumference of each.

Ans. Fore wheel 3 yards, and hind wheel 4 yards.

SECTION XIII.

RATIO AND PROPORTION.

(174.) NUMBERS may be compared in two ways, either by means of their *difference*, or by their *quotient*. We may inquire *how much* one quantity is greater than another, or *how many times* the one contains the other. One is called Arithmetical, and the other Geometrical Ratio.

(175.) The difference between two numbers is called their *Arithmetical Ratio*. Thus the arithmetical ratio of 9 to 7 is 9−7 or 2; and if a and b designate two numbers, their arithmetical ratio is designated by $a-b$.

(176.) Numbers are more generally compared by means of quotients; that is, by inquiring how many times one number contains another. The quotient of one number divided by another is called their *Geometrical Ratio*. The term Ratio, when used without any qualification, is always understood to signify a geometrical ratio; and we shall confine our attention to ratios of this description.

(177.) By the *ratio* of two numbers, then, we mean *the quotient which arises from dividing one of these numbers by the other*.

Thus the ratio of 12 to 4 is represented by $\frac{12}{4}$ or 3.

QUEST.—In how many ways may numbers be compared? What is Arithmetical Ratio? What is Geometrical Ratio?

If a and b represent two quantities of the same kind, the ratio of a to b is the quotient arising from dividing a by b, and may be represented by writing them $a:b$, or $\frac{a}{b}$. The first term, a, is called the *antecedent* of the ratio; the last term, b, is called the *consequent* of the ratio.

Hence it appears that the theory of ratios is included in the theory of fractions; and a ratio may be considered as *a fraction whose numerator is the antecedent, and whose denominator is the consequent.*

(178.) *Proportion is an equality of ratios.*

Thus, if we take four numbers,

$$3, 4, 9, 12,$$

such that the quotient of the first, divided by the second, is equal to the quotient of the third divided by the fourth, the numbers are said to be proportional, and the proportion may be written

$$\frac{3}{4}=\frac{9}{12},$$

or

$$3:4::9:12.$$

In general, if a, b, c, d are four quantities such that a, when divided by b, gives the same quotient as c when divided by d, then a, b, c, d are called *proportionals*, and we say that a is to b as c is to d; and this is expressed by writing them thus:

$$a:b::c:d,$$

or

$$a:b=c:d,$$

or

$$\frac{a}{b}=\frac{c}{d}.$$

QUEST.—Define the terms antecedent and consequent. What is a proportion?

(179.) In ordinary language, the terms *ratio* and *proportion* are confounded with each other. Thus two quantities are said to be in the proportion of 3 to 5 instead of the ratio of 3 to 5. In strictness, however, a ratio subsists between *two* quantities, a proportion only between *four*. Ratio is the quotient arising from dividing one quantity by another; *two equal ratios form a proportion.*

(180.) In the proportion

$$a:b::c:d,$$

a, b, c, d are called the *terms* of the proportion. The first and last terms are called the *extremes*, the second and third the *means*. The first term is called the *first antecedent*, the second term the *first consequent*, the third term the *second antecedent*, and the fourth term the *second consequent*. The last term is said to be a fourth proportional to the other three taken in order.

(181.) The word *term*, when applied to a proportion, is used in a slightly different sense from that explained in Art. 12. The terms of a proportion may be polynomials. Thus

$$a+b:c+d::e+f:g+h.$$

(182.) Three quantities are said to be in proportion when the first has the same ratio to the second that the second has to the third, and then the middle term is said to be a *mean proportional* between the other two. For example,

$$2:4::4:8,$$

where 4 is a mean proportional between 2 and 8.

QUEST.—Explain the difference between a ratio and a proportion? How are the terms of a proportion distinguished? When are three quantities said to be proportional?

(183.) If four quantities are proportional, *the product of the two extremes is equal to the product of the two means.*

Let $a:b::c:d;$

then will $ad=bc.$

For, since the four quantities are proportional,

$$\frac{a}{b}=\frac{c}{d},$$

and, by clearing the equation of fractions, we have

$$ad=bc.$$

Thus, if $3:4::9:12,$

then $3\times12=4\times9.$

(184.) Conversely, if the product of two quantities is equal to the product of two others, *the first two quantities may be made the extremes, and the other two the means of a proportion.*

Let $ad=bc;$

then will $a:b::c:d.$

For, since $ad=bc,$

dividing each of these equals by bd the expression becomes

$$\frac{a}{b}=\frac{c}{d};$$

that is, $a:b::c:d.$

Thus, if $3\times12=4\times9,$

then $3:4::9:12.$

(185.) The preceding proposition is called the *test* of proportions, and any change may be made in the form of a proportion which is consistent with the ap-

QUEST.—If four quantities are proportional, by what property may they be distinguished? How may every equation be converted into a proportion?

plication of this test. In order, then, to decide whether four quantities are proportional, we must compare the product of the extremes with the product of the means.

Thus, to determine whether the numbers 5, 6, 7, 8 are proportional, we multiply 5 by 8, and obtain 40. Multiplying 6 by 7, we obtain 42. As these two products are *not* equal, we conclude that the numbers 5, 6, 7, 8 are *not proportional.*

Again: take the numbers 5, 6, 10, 12. The product of 5 by 12 is 60, and the product of 6 by 10 is also 60. Hence these numbers are proportional; that is,

$$5:6::10:12.$$

(186.) If three quantities are in continued proportion, *the product of the extremes is equal to the square of the mean.*

If $a:b::b:c$,

then, by Art. 183, $ac=bb$, which is equal to b^2.

Thus, if $3:6::6:12$,

then $3\times12=6^2$.

(187.) Conversely, if the product of two quantities is equal to the square of a third, *the last quantity is a mean proportional between the other two.*

Thus, let $ac=b^2$.

Dividing these equals by bc, we obtain

$$\frac{a}{b}=\frac{b}{c},$$

or $a:b::b:c$.

QUEST.—How may we determine whether four quantities are proportional? When three quantities are in continued proportion, by what property are they distinguished? How is a mean proportional between two quantities found?

Thus, if $4\times 9=6^2$,
then 6 is a mean proportional between 4 and 9.

Examples.

1. Given the first three terms of a proportion, 24, 15, and 40, to find the fourth term.

2. Given the first three terms of a proportion, $3ab^3$, $4a^2b^2$, and $9a^3b$, to find the fourth term.

3. Given the last three terms of a proportion, $4a^2b^5$, $3a^3b^3$, and $2a^5b$, to find the first term.

4. Given the first, second, and fourth terms of a proportion, $5y^4$, $7x^2y^3$, and $21x^6y$, to find the third term.

5. Given the first, third, and fourth terms of a proportion, 22, 72, and 252, to find the second term.

6. Are the quantities 25, 70, 78, and 218 proportional?

7. Resolve the equation $22\times 105=33\times 70$ into a proportion.

(188.) *Ratios that are equal to the same ratio are equal to each other.*

Let $a:b::x:y,$
and $c:d::x:y,$ } then will $a:b::c:d$.

For, since $a:b::x:y,$

we have $\frac{a}{b}=\frac{x}{y}.$

And since $c:d::x:y,$

we have $\frac{c}{d}=\frac{x}{y}.$

Therefore $\frac{a}{b}=\frac{c}{d},$

and hence $a:b::c:d.$

QUEST.—Compare two ratios that are equal to the same ratio.

Thus, if $2:7::18:63$,
and $2:7::22:77$,
then $18:63::22:77$.

(189.) If four quantities are proportional, they will be proportional by *alternation;* that is, *the first will have the same ratio to the third that the second has to the fourth.*

Let $a:b::c:d$
then will $a:c::b:d$.
For, since $a:b::c:d$,
by Art. 183, $ad=bc$;
and since $ad=bc$,
by Art. 184, $a:c::b:d$.
Thus, if $4:6::28:42$,
then, by alternation,
$4:28::6:42$.

(190.) If four quantities are proportional, they will be proportional by *inversion;* that is, *the second will have to the first the same ratio that the fourth has to the third.*

Let $a:b::c:d$;
then will $b:a::d:c$.
For, since $a:b::c:d$,
by Art. 183, $ad=bc$,
or $bc=ad$.
Therefore, by Art. 184,
$b:a::d:c$.
Thus, if $5:12::15:36$,
then $12:5::36:15$.

QUEST.—Explain the principle of alternation. Explain the principle of inversion.

(191.) If four quantities are proportional, they will be proportional by *composition;* that is, *the sum of the first and second will have to the second the same ratio that the sum of the third and fourth has to the fourth.*

Let $a:b::c:d;$
then will $a+b:b::c+d:d.$

For, since $a:b::c:d,$
we have $\frac{a}{b}=\frac{c}{d}.$

Add unity to each of these equals, and we have

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d};$$

that is, $a+b:b::c+d:d.$

Thus, if $7:11::21:33,$
then $7+11:11::21+33:33;$
that is, $18:11::54:33.$

(192.) If four quantities are proportional, they will be proportional by *division;* that is, *the difference of the first and second will have to the second the same ratio that the difference of the third and fourth has to the fourth.*

Let $a:b::c:d;$
then will $a-b:b::c-d:d.$

For, since $a:b::c:d,$
we have $\frac{a}{b}=\frac{c}{d}.$

Subtract unity from each of these equals, and we have

QUEST.—Explain the principle of composition. Explain the principle of division.

$$\frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d};$$

that is, $a-b:b::c-d:d.$

Thus, if $11:7::33:21,$

then $11-7:7::33-21:21;$

that is, $4:7::12:21.$

(193.) *Equal multiples of two quantities have the same ratio as the quantities themselves.*

The ratio of a to b is represented by the fraction $\frac{a}{b}$; and the value of a fraction is not changed if we multiply or divide both numerator and denominator by the same quantity. Thus,

$$\frac{a}{b}=\frac{ma}{mb},$$

or $a:b::ma:mb.$

Thus, $5:7::5m:7m.$

(194.) If there is any number of proportional quantities all having the same ratio, *the first will have to the second the same ratio that the sum of all the antecedents has to the sum of all the consequents.*

Let a, b, c, d, e, f be any number of proportional quantities, such that

$$a:b::c:d::e:f,$$

then will $a:b::a+c+e:b+d+f.$

For, since $a:b::c:d,$

we have $ad=bc.$

And since $a:b::e:f,$

we have $af=be.$

QUEST.—Compare equal multiples of two quantities. What principle may be applied to any number of proportional quantities all having the same ratio?

To these equals add $ab=ba$,
and we obtain $a(b+d+f)=b(a+c+e)$.

Hence, by Art. 184,

$$a:b::a+c+e:b+d+f.$$

Thus, if we have

$$2:3::4:6::8:12;$$

then $$2:3::2+4+8:3+6+12,$$

or $$2:3::14:21.$$

(195.) If four quantities are proportional, *like powers or roots of these quantities will also be proportional.*

Let $$a:b::c:d;$$

then will $$a^2:b^2::c^2:d^2;$$

and $$a^3:b^3::c^3:d^3, \text{ etc.}$$

For, since $$a:b::c:d,$$

we have $$\frac{a}{b}=\frac{c}{d}.$$

Squaring each of these equal quantities, we have

$$\frac{a^2}{b^2}=\frac{c^2}{d^2};$$

that is, $$a^2:b^2::c^2:d^2;$$

and the same may be proved of the cube or any other power.

Thus, if we have

$$2:3::4:6;$$

then $$2^2:3^2::4^2:6^2;$$

that is, $$4:9::16:36.$$

(196.) If there are two sets of proportional quantities, *the products of the corresponding terms will be proportional.*

QUEST.—Compare like powers or roots of proportional quantities. What principle may be applied to two sets of proportional quantities?

Let $a:b::c:d$,
and $e:f::g:h$;
then will $ae:bf::cg:dh$.

For, since $a:b::c:d$,
by Art. 183, $ad=bc$;
and since $e:f::g:h$,
by Art. 183, $eh=fg$.

Multiplying these equals together, we have

$$ae\times dh=bf\times cg.$$

Hence, by Art. 184,

$$ae:bf::cg:dh.$$

Thus, if we have $2:3::4:6$,
and $5:7::10:14$;
then $2\times5:3\times7::4\times10:6\times14$,
or $10:21::40:84$.

SECTION XIV.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

(197.) *An Arithmetical Progression is a series of quantities which increase or decrease by the continued addition or subtraction of the same quantity.*

Thus the numbers

1, 3, 5, 7, 9, 11, etc.,

which are obtained by the addition of 2 to each successive term, form what is called an *increasing* Arithmetical Progression; and the numbers

20, 17, 14, 11, 8, 5, etc.,

which are obtained by the subtraction of 3 from each successive term, form what is called a *decreasing* Arithmetical Progression.

PROBLEM I.

(198). *To find any term of an Arithmetical Progression.*

If a represent the first term of an increasing arithmetical progression, and d the common difference, the second term of the series will be $a+d$, the third $a+2d$, the fourth $a+3d$, the fifth $a+4d$, etc.

QUEST.—What is an Arithmetical Progression? What is an increasing Progression? What is a decreasing Progression? How may we find any term of an Arithmetical Progression?

The coefficient of d in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on; that is, any term of the series is equal to the first term, plus as many times the common difference as there are preceding terms.

If we represent any term of the series by l, and suppose n to be the number which marks the place of that term in the series, the expression for this term will be

$$l=a+(n-1)d.$$

(199.) Hence, if we put l to represent the last term of the series, we shall have the following

RULE.

The last term of an increasing arithmetical progression is equal to the first term, plus the product of the common difference into the number of terms less one.

This rule enables us to find any term of a series without being obliged to determine all those which precede it.

Examples.

1. What is the fourth term of a series whose first term is 3 and common difference 2?

Ans. 9.

2. What is the sixth term of a series whose first term is 5 and common difference 3?

Ans. 20.

QUEST.—Give the rule for finding the last term of an increasing arithmetical progression.

3. What is the eighth term of a series whose first term is 7 and common difference 4?

Ans. 35.

4. What is the tenth term of a series whose first term is 9 and common difference 5?

Ans. 54.

5. What is the twelfth term of a series whose first term is 11 and common difference 6?

Ans. 77.

6. What is the twentieth term of a series whose first term is 13 and common difference 7?

Ans. 146.

7. What is the thirtieth term of a series whose first term is 15 and common difference 8?

Ans. 247.

8. What is the fortieth term of a series whose first term is 20 and common difference 9?

Ans. 371.

9. What is the fiftieth term of the series

1, 6, 11, 16, 21, etc.?

Ans. 246.

10. What is the hundredth term of the series

1, 7, 13, 19, 25, etc.?

Ans. 595.

(200.) If a represent the first term of a decreasing arithmetical progression, and d the common difference, the second term of the series will be $a-d$, the third $a-2d$, the fourth $a-3d$, etc., and the expression for any term of the series will be

$$l=a-(n-1)d.$$

Hence, to find the last term of a decreasing arithmetical progression, we have the following

Rule.

The last term of a decreasing arithmetical progression is equal to the first term, minus the product of the common difference into the number of terms less one.

Examples.

1. If the first term of a decreasing progression is 80, the number of terms 15, and the common difference 5, what is the last term?

Ans. $l = a - (n-1)d = 80 - 14 \times 5 = 10$.

2. What is the twentieth term of a series whose first term is 53 and common difference 2?

Ans. 15.

3. What is the thirtieth term of a series whose first term is 114 and common difference 3?

Ans. 27.

4. What is the fiftieth term of a series whose first term is 228 and common difference 4?

Ans. 32.

5. What is the hundredth term of a series whose first term is 648 and common difference 6?

Ans. 54.

Problem II.

(201.) *To find the sum of the terms of an arithmetical series.*

Take any arithmetical series, and under it set the same terms in an inverted order thus:

Quest.—How may we find the last term of a decreasing arithmetical progression? How may we find the sum of the terms of an arithmetical series?

Let the series be	1, 3, 5, 7, 9, 11, 13, 15;
the same series inverted is	15, 13, 11, 9, 7, 5, 3, 1.
The sums are	16, 16, 16, 16, 16, 16, 16, 16.

The sum of all the terms in the *double* series is equal to the sum of the extremes 1 and 15, repeated as many times as there are terms, that is, 8 times; and this is double the sum of the terms of a *single* series. Hence the sum of the terms of the proposed series is equal to

$$\frac{8(15+1)}{2}=64.$$

(202.) In order to generalize this method, put S to represent the sum of the terms of the series

$$a,\ a+d,\ a+2d, \text{ etc.},$$

continued to l, which we employ to represent the last term; that is,

$$S=a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+\ .\ .\ .\ .\ .\ +l.$$

Under it write the same series in an inverted order thus:

$$S=l+\overline{l-d}+\overline{l-2d}+\overline{l-3d}+\ .\ .\ .\ .\ .\ .\ +a.$$

If we add together the corresponding terms of the two series, we shall obtain

$$2S=\overline{l+a}+\overline{l+a}+\overline{l+a}+\overline{l+a}+\ .\ .\ .\ +\overline{l+a}.$$

If we represent the number of terms of the series by n, we shall have

$$2S=n(l+a);$$

whence

$$S=\frac{n(l+a)}{2}.$$

(203.) Hence we derive the following

Rule.

The sum of the terms of an arithmetical progression is equal to half the sum of the two extremes, multiplied by the number of terms.

We also see from the preceding demonstration, that *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

Examples.

1. What is the sum of the natural series of numbers 1, 2, 3, 4, 5, etc., up to 25?

Ans. $S=\frac{n(l+a)}{2}=\frac{25\times(25+1)}{2}=325.$

2. The extremes of an arithmetical progression are 2 and 50, and the number of terms 17. What is the sum of the series?

Ans. 442.

3. The extremes of an arithmetical progression are 10 and 20, and the number of terms 6. What is the sum of the series?

Ans. 90.

4. The extremes of an arithmetical progression are 3 and 19, and the number of terms 9. What is the sum of the series?

Ans. 99.

5. The extremes of an arithmetical progression are 5 and 595, and the number of terms 60. What is the sum of the series?

Ans. 18000.

QUEST.—Give the rule for finding the sum of the terms of an arithmetical progression. Explain the reason of the rule.

6. The extremes of an arithmetical progression are 5 and 92, and the number of terms 30. What is the sum of the series?

Ans. 1455.

7. What is the sum of 100 terms of the series

$$1, 3, 5, 7, 9, \text{etc.}?$$

Ans. 10000.

(204.) If we take the equation

$$l=a+(n-1)d,$$

and transpose the term $(n-1)d$, we obtain

$$a=l-(n-1)d;$$

that is, *the first term of an increasing arithmetical progression is equal to the last term, minus the product of the common difference by the number of terms less one.*

(205.) If we transpose the term a, and divide by $n-1$, we obtain

$$d=\frac{l-a}{n-1};$$

that is, in an arithmetical progression, *the common difference is equal to the difference between the two extremes, divided by the number of terms less one.*

Examples

1. The last term of a progression is 48, the first term 6, and the number of terms 15. What is the common difference?

Ans. 3.

QUEST.—How may we find the first term of an increasing progression? How may we find the common difference in an arithmetical progression?

2. The last term of a progression is 68, the first term 8, and the number of terms 16. What is the common difference?

Ans. 4.

3. The last term of a progression is 105, the first term 10, and the number of terms 20. What is the common difference?

Ans. 5.

PROBLEM III.

(206.) *To find any number of arithmetical means between two given numbers.*

In order to solve this problem, we must first find the common difference. The whole number of terms of the series consists of the two extremes and all the intermediate terms. If, then, m represent the required number of means, $m+2$ will be the whole number of terms.

By substituting $m+2$ for n in the formula, Art. 205, we obtain

$$d=\frac{l-a}{m+1};$$

that is, *the common difference of the required series is equal to the difference between the two given numbers, divided by the number of means plus one.*

Having obtained the common difference, the required means are easily obtained by addition.

Ex. 1. Find 9 arithmetical means between 2 and 32.

$$d=\frac{l-a}{m+1}=\frac{32-2}{10}=3, \text{ the common difference.}$$

QUEST.—How may we find any number of arithmetical means between two numbers?

Hence the progression is

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

Ex. 2. Find 10 arithmetical means between 3 and 58.

Ans. 8, 13, 18, 23, 28, 33, 38, 43, 48, and 53.

Ex. 3. Find 6 arithmetical means between 1 and 50.

Ans. 8, 15, 22, 29, 36, and 43.

(207.) *Examples.*

1. A student bought 7 books, the prices of which were in arithmetical progression. The price of the cheapest was 5 shillings, and the price of the dearest 23 shillings. What was the price of each book?

Ans. 5, 8, 11, 14, 17, 20, and 23 shillings.

2. What is the nth term of the series

1, 3, 5, 7, 9, etc.?

Ans. $2n-1$;

that is, *the last term of this series is one less than twice the number of terms.*

3. What is the sum of n terms of the series

1, 3, 5, 7, 9, etc.?

Ans. n^2;

that is, *the sum of the terms of this series is equal to the square of the number of terms.*

Thus,
$$1+3 = 4=2^2,$$
$$1+3+5 = 9=3^2,$$
$$1+3+5+7 =16=4^2,$$
$$1+3+5+7+9=25=5^2.$$

4. What is the sum of n terms of the series

QUEST.—What is the last term of the series of odd numbers beginning with unity? What is the sum of the series of odd numbers beginning with unity?

1, 2, 3, 4, 5, etc.?

Ans. $\frac{n(n+1)}{2}$.

5. What is the sum of n terms of the series
2, 4, 6, 8, 10, etc?

Ans. $n(n+1)$

6. One hundred stones being placed on the ground in a straight line, at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket which is placed two yards from the first stone?

Ans. 20,200 yards.

GEOMETRICAL PROGRESSION.

(208.) *A Geometrical Progression is a series of quantities, each of which is equal to the product of that which precedes it by a constant number.*

Thus the series

2, 4, 8, 16, 32, etc.,

and 81, 27, 9, 3, etc.,

are geometrical progressions. In the former, each number is derived from the preceding by multiplying it by 2, and the series forms an *increasing* geometrical progression. In the latter, each number is derived from the preceding by multiplying it by $\frac{1}{3}$, and the series forms a *decreasing* geometrical progression.

In each of these cases the common multiplier is called the *common ratio.*

QUEST.—What is a Geometrical Progression? What is an increasing progression? What is a decreasing progression? What is the common ratio?

PROBLEM IV.

(209.) *To find any term of a geometrical progression.*

If a represent the first term of the progression, and r the common ratio, the second term of the series will be ar, the third ar^2, the fourth ar^3, the fifth ar^4, etc.

The exponent of r in the *second* term is 1, in the *third* term is 2, in the *fourth* term is 3, and so on; hence the exponent of r in the nth term of the series will be $n-1$; that is, the nth term of the series may be written

$$ar^{n-1}.$$

If we represent any term of the series by l, and suppose n to be the number which marks the place of that term in the series, the expression for this term will be

$$l=ar^{n-1}.$$

(210.) Hence, if we put l to represent the last term of the series, we shall have the following

RULE.

The last term of a geometrical progression is equal to the product of the first term by that power of the ratio whose exponent is one less than the number of terms.

This rule will enable us to find any term of a series without being obliged to determine all those which precede it.

QUEST.—How may we find any term of a geometrical progression? Give the rule for the last term of a geometrical progression. Explain the reason of the rule.

Examples.

1. What is the sixth term of a geometrical progression whose first term is 3 and common ratio 2?

Ans. $l = ar^{n-1} = 3 \times 2^5 = 3 \times 32 = 96$.

2. What is the seventh term of a geometrical progression whose first term is 4 and common ratio 3?

Ans. 2916.

3. What is the eighth term of a progression whose first term is 5 and common ratio 4?

Ans. 81920.

4. What is the ninth term of a progression whose first term is 6 and common ratio 3?

Ans. 39366.

5. What is the tenth term of a progression whose first term is 7 and common ratio 2?

Ans. 3584.

6. What is the ninth term of the series

1, 3, 9, 27, 81, etc.?

Ans. 6561.

7. What is the eighth term of the series

1, 4, 16, 64, 256, etc.?

Ans. 16384.

8. What is the seventh term of the series

1, 5, 25, 125, 625, etc.?

Ans. 15625.

PROBLEM V.

(211.) *To find the sum of the terms of a geometrical progression.*

If we take any geometrical series, and multiply

QUEST.—How may we find the sum of the terms of a geometrical progression?

each of its terms by the ratio, a new series will be formed, of which every term, except the last, will have its corresponding term in the first series. Thus, take the series

$$1, 3, 9, 27, 81,$$

the sum of which we will represent by S, so that

$$S=1+3+9+27+81.$$

Multiplying each term by 3, we obtain

$$3S=3+9+27+81+243.$$

The terms of the two series are identical, except the *first* term of the first series and the *last* term of the second series. If then we subtract one of these equations from the other, all the remaining terms will disappear, and we shall have

$$3S-S=243-1,$$

or $$S=\frac{243-1}{3-1}.$$

(212.) In order to generalize this method, let a, ar, ar^2, etc., represent any geometrical series, the last term of which is l, and let S represent the sum of all the terms; then

$$S=a+ar+ar^2+ar^3+\ .\ .\ .\ .\ .\ .\ .\ +l.$$

Multiplying this equation by r, we obtain

$$rS=ar+ar^2+ar^3+ar^4+\ .\ .\ .\ .\ .\ +lr.$$

Subtracting the first equation from the second, all the terms in the second members disappear, except the first term of the first series and the last term of the second series, and we obtain

$$rS-S=lr-a;$$

whence $$S=\frac{lr-a}{r-1}.$$

(213.) Hence, to find the sum of the terms of a geometrical progression, we have the following

RULE.

Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.

Examples.

1. What is the sum of nine terms of the series

1, 3, 9, 27, 81, etc.?

We have already found the ninth term of the series to be 6561.

Hence $S=\frac{lr-a}{r-1}=\frac{6561\times3-1}{3-1}=\frac{19682}{2}=9841.$

2. What is the sum of eight terms of the series

1, 4, 16, 64, etc.?

Ans. 21845.

3. What is the sum of 14 terms of the series

1, 2, 4, 8, 16, etc.?

Ans. 16383.

4. The extremes of a geometrical progression are 3 and 12288, and the common ratio 2. What is the sum of the series?

Ans. 24573.

5. The extremes of a geometrical progression are 4 and 78732, and the common ratio 3. What is the sum of the series?

Ans. 118096.

6. What debt may be discharged in 12 months by

QUEST.—Give the rule for the sum of the terms of a geometrical progression. Explain the reason of the rule.

paying one dollar the first month, two dollars the second month, four dollars the third; and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. the debt is \$4095,
the last payment \$2048.

(214.) When the progression is a decreasing one, and r consequently represents a fraction, the expression for the sum of the series is written

$$S=\frac{a-lr}{1-r},$$

in order that both terms of the fraction may be positive.

Ex. 1. What is the sum of 9 terms of the progression
1536, 768, 384, etc.?

Here $l=1536\times(\frac{1}{2})^8=6$. Hence $S=\dfrac{1536-3}{\frac{1}{2}}=3066$.

Ex. 2. What is the sum of 11 terms of the progression 5120, 2560, 1280, etc.?

Ans. 10235.

Ex. 3. What is the sum of 12 terms of the progression 8192, 4096, 2048, etc.?

Ans. 16380.

Ex. 4. What is the sum of 7 terms of the progression
15625, 3125, 625, etc.?

Ans. 19531.

Ex. 5. What is the sum of 8 terms of the progression
32768, 8192, 2048, etc.?

Ans. 43690.

Ex. 6. What is the sum of 9 terms of the progression
19683, 6561, 2187, etc.?

Ans. 29523.

QUEST.—What is the sum of the terms of a decreasing progression?

PROBLEM VI.

(215.) *To find a mean proportional between two numbers.*

According to Art. 187, if b is a mean proportional between a and c, we shall have

$$b^2=ac;$$

and hence $$b=\sqrt{ac};$$

that is, to find a mean proportional between two numbers, *multiply the two numbers together, and extract the square root of their product.*

Ex. 1. What is the geometrical mean between the numbers 4 and 9?

Ans. $\sqrt{4\times9}=6$.

Ex. 2. What is the geometrical mean between the numbers 4 and 25?

Ans. 10.

Ex. 3. What is the geometrical mean between the numbers 9 and 16?

Ans. 12.

Ex. 4. What is the geometrical mean between the numbers 4 and 49?

Ans. 14.

Ex. 5. What is the geometrical mean between the numbers 9 and 25?

Ans. 15.

Ex. 6. What is the geometrical mean between the numbers 4 and 81?

Ans. 18.

QUEST.—How may we find a mean proportional between two numbers?

Ex. 7. What is the geometrical mean between the numbers 16 and 25?

Ans. 20.

(216.) *Of decreasing progressions having an infinite number of terms.*

In a decreasing progression, the expression for the sum of the series is

$$S=\frac{a-lr}{1-r}.$$

For l substitute its value ar^{n-1}, and we obtain

$$S=\frac{a-ar^n}{1-r},$$

which may be written

$$S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

In a decreasing progression, since r is less than unity, r^n is less than unity; and the larger the number n, the smaller will be the quantity r^n. If, therefore, we take a very large number of terms of the series, the quantity r^n, and consequently the term $\frac{ar^n}{1-r}$, will be very small; and if we take n greater than any assignable number, then $\frac{ar^n}{1-r}$ will be less than any assignable number, and may be neglected; that is, when the number of terms is infinite, we have

$$S=\frac{a}{1-r}.$$

(217.) Hence the sum of an infinite series decreasing in geometrical progression is found by the following

Rule.

Divide the first term by unity diminished by the ratio.

Ex. 1. Find the sum of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+, \text{ etc.}$$

Here $a=1$, $r=\frac{1}{2}$.

Therefore $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2.$

Ex. 2. Find the sum of the infinite series

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+, \text{ etc.}$$

Ans. $\frac{3}{2}$.

Ex. 3. Find the sum of the infinite series

$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+, \text{ etc.}$$

Ans. $\frac{4}{3}$.

Ex. 4. Find the sum of the infinite series

$$1+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+, \text{ etc.}$$

Ans. $\frac{5}{4}$.

Ex. 5. Find the sum of the infinite series

$$1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+, \text{ etc.}$$

Ans. $\frac{10}{9}$.

Quest.—Give the rule for the sum of an infinite decreasing series. Explain the reason of the rule.

(218.) MISCELLANEOUS PROBLEMS.

Prob. 1. Find a number such that one third thereof, increased by one fourth of the same, shall be equal to one sixth of it, increased by 30.

Ans. 72.

Prob. 2. Divide \$1340 among three persons, A, B, and C, so that B may receive \$100 more than A, and C \$180 more than B. How much should each receive?

Ans. A \$320, B \$420, C \$600.

Prob. 3. In a mixture of wine and cider, one half of the whole, plus 21 gallons, was wine; and one third part, minus 6 gallons, was cider. How many gallons were there of each?

Ans. 66 of wine and 24 of cider.

Prob. 4. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 120 years. What is the age of each?

Ans. A's is 72, B's is 36, and C's is 12 years.

Prob. 5. Two persons, A and B, lay out equal sums of money in trade; A gains \$504, and B loses \$348, and A's money is now double of B's. What sum did each lay out?

Ans. \$1200.

Prob. 6. A gentleman bought a chaise, horse, and harness for \$315. The horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness. What did he give for each?

Ans. \$210 for the chaise, \$70 for the horse, and \$35 for the harness.

Prob. 7. Two persons, A and B, have both the same income. A saves one fifth of his yearly; but B, by spending $500 per year more than A, at the end of 4 years finds himself $1000 in debt. What was his income?

Ans. $1250.

Prob. 8. A person in play lost one fourth of his money, and then won $3; after which he lost one third of what he then had, and then won $2. Lastly, he lost one seventh of what he then had, and found he had but $12 remaining. What had he at first?

Ans. $20.

Prob. 9. A person goes to a tavern with a certain sum of money in his pocket, where he spends 8 shillings. He then borrows as much money as he had left, and going to another tavern, he there spends 8 shillings also. Then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 8 shillings, and borrowed as much as he had left; and again spending 8 shillings at a fourth tavern, he then had nothing remaining. What had he at first?

Ans. 15 shillings.

Prob. 10. A father gives to his five sons $950, which they are to divide according to their ages, so that each elder son shall receive $20 more than his next younger brother. What is the share of the youngest?

Ans. $150.

Prob. 11. A gentleman has two horses and two saddles, one of which cost $30, the other $5. If he places the best saddle upon the first horse, and the worst upon the second, then the latter is worth $5 more than the other. But if he puts the worse saddle upon the first horse, and the best upon the other, then the latter is worth twice as much as the first. What is the value of each horse?

Ans. The first $50, the second $80.

Prob. 12. There are two numbers whose sum is 37, and if three times the less be subtracted from four times the greater, and this difference be divided by 6, the quotient will be 6. What are the numbers?

Ans. 16 and 21.

Prob. 13. Find three numbers such that the first, with half the sum of the second and third, shall be 120; the second, with one fifth the difference of the third and first, shall be 70; and half the sum of the three numbers shall be 95.

Ans. 50, 65, and 75.

Prob. 14. A banker has 2640 coins of two kinds, and there are four and a half times as many of one sort as of the other. How many has he of each sort?

Ans. 480 and 2160.

Prob. 15. Divide the number a into two such parts that one may be m times as great as the other.

Ans. $\frac{a}{m+1}$ and $\frac{ma}{m+1}$.

Prob. 16. How much money have I in my pocket when the fourth and fifth parts of the same together amount to $9?

Ans. $20.

Prob. 17. Divide the number 46 into two parts, so that when the one is divided by 7, and the other by 3, the quotients together may amount to 10.

Ans. 28 and 18.

Prob. 18. A fortress has a garrison of 2600 men, among whom there are 9 times as many foot soldiers, and 3 times as many artillery soldiers, as cavalry. How many of each corps are there?

Ans. 200 cavalry, 600 artillery, and 1800 foot.

Prob. 19. I have a certain number in my thoughts, says A to B; try to guess it. I multiply it by 7, add 3 to the product, divide this by 2, subtract 4 from the quotient, and obtain 15. What number is it?

Ans. 5.

Prob. 20. There are three numbers such that the second, divided by the first, gives 2 for a quotient and 1 for a remainder; while the third, divided by the second, gives 3 for a quotient with the remainder 3. The sum of these three numbers is 70. What are the numbers?

Ans. 7, 15, and 48.

Prob. 21. An arithmetician desires his scholars to find a number which he has in his mind from the following data. If, says he, you multiply the number by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, you will obtain this same number. What number is it?

Ans. 54.

Prob. 22. Two purses together contain $300. If we take $30 out of the first and put it into the second, there will be the same sum in each. How much does each contain?

Ans. The first $180, the second $120

Prob. 23. A says to B, give me $100, and then I shall have as much money as you. No, says B to A, give me rather $100, and then I shall have twice as much as you. How much has each?

Ans. A $500 and B $700.

Prob. 24. Find two numbers of the following properties. When the one is multiplied by 2, the other by 5, and both products added together, the sum is 31; on the other hand, if the first be multiplied by 7, the second by 4, and both products added together, we obtain 68.

Ans. The first is 8, the second is 3.

Prob. 25. In the composition of a certain quantity of gunpowder, the nitre was ten pounds more than two thirds of the whole; the sulphur was four and a half pounds less than one sixth of the whole; and the charcoal was two pounds less than one seventh of the nitre. How many pounds of gunpowder were there?

Ans. 69 pounds.

Prob. 26. Two workmen received the same sum for their labor; but if one had received $24 more, and the other $16 less, then one would have received just three times as much as the other. What did each receive?

Ans. $36.

Prob. 27. Find two numbers of which the less is to the greater as 2 to 3, and whose product is twelve times the sum of the numbers.

Ans. 20 and 30.

Prob. 28. There are two numbers whose difference is 12; and if eight times the less be subtracted from five times the greater, the remainder will be 6. What are the numbers?

Ans. 18 and 30.

Prob. 29. What number is that, one third, one fourth, and two fifths of which being added together will make 59?

Ans. 60.

Prob. 30. A grocer has two kinds of tea, one of which is worth 57 cents per pound, the other 81 cents. How many pounds of each must he take to form a chest of 104 pounds, which will be worth $67.20?

Ans. 71 pounds of the former and 33 of the latter.

Prob. 31. There is a number consisting of two digits, whose sum is 7; and if 9 be added to the number itself, the digits will be inverted? What is the number?

Ans. 34.

Prob. 32. Out of a hogshead of water which had leaked away one third part, 21 gallons were drawn, and then being gauged, it was found to be half full. How many gallons did it hold?

Ans. 126.

Prob. 33. A pile stands one third in the ground, one half in the water, and three feet above the water. What is its entire length?

Ans. 18 feet.

Prob. 34. A and B start at the same time and place to go round an island 600 miles in circumference. A goes 30 miles a day, and B 20. How long before they will both be at the starting point together, and how far will each have traveled?

Ans. In 60 days; and A will have gone three times round the island, and B twice.

Prob. 35. A farmer sold 5 bushels of wheat and 6 of corn for 54 shillings. He afterward sold 4 bushels of wheat and 3 of corn for 36 shillings. What was the price of each per bushel?

Ans. Wheat 6 shillings and corn 4 shillings.

Prob. 36. Two persons, A and B, agree together to purchase a house worth $1800. Says A to B, give me two thirds of your money, and I can purchase it alone; but says B to A, if you give me three fourths of your money, I shall be able to purchase it alone. How much had each?

Ans. A had $1200 and B $900.

Prob. 37. A father directs that $1125 be divided among his three sons, in proportion to their ages. The eldest is twice as old as the youngest, and the age of the second is one half greater than that of the youngest. How much should each receive?

Ans. The youngest $250, the second $375, and the eldest $500.

Prob. 38. Three regiments are to furnish 603 men, and each to furnish in proportion to its strength. Now the strength of the first is to that of the second as 3 to 5, and the second is to the third as 4 to 7. How many must each furnish?

Ans. The first 108, the second 180, and the third 315 men.

Prob. 39. Divide the number 60 into two such parts that their product may be to the sum of their squares in the ratio of 2 to 5.

Ans. 20 and 40.

Prob. 40. Find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21.

Ans. 7 or 3.

Prob. 41. A merchant buys several pieces of cloth at the same price for $60. If he had bought three pieces more for the same sum, each piece would have cost him one dollar less. How many pieces did he buy?

Ans. 12.

Prob. 42. What number is that whose half, multiplied by its third part, produces 1176?

Ans. 84.

Prob. 43. The difference of two numbers is 6, and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

Ans. 17 and 11, or 7 and 1.

Prob. 44. What number is that whose seventh and eighth parts multiplied together, and the product divided by 3, gives the quotient 672?

Ans. 336.

Prob. 45. Find two numbers such that their product shall be 675; and the quotient of the greater, divided by the less, shall be 3.

Ans. 45 and 15.

Prob. 46. There are two numbers, one of which exceeds the other by 8, and whose product is 240. What are the numbers?

Ans. 12 and 20.

Prob. 47. Find two numbers in the ratio of 3 to 5, and whose product is 375.

Ans. 15 and 25.

Prob. 48. It is required to find two numbers whose product shall be 48, and the difference of their squares 28.

Ans. 6 and 8.

Prob. 49. What two numbers are those whose sum is 41, and the sum of whose squares is 901?

Ans. 26 and 15.

Prob. 50. What two numbers are those whose sum is equal to a, and the sum of whose squares is equal to b?

Ans. $\frac{a+\sqrt{2b-a^2}}{2}$, and $\frac{a-\sqrt{2b-a^2}}{2}$.

Prob. 51. What two numbers are as m to n, and the sum of whose squares is equal to b?

Ans. $\frac{m\sqrt{b}}{\sqrt{m^2+n^2}}$, $\frac{n\sqrt{b}}{\sqrt{m^2+n^2}}$.

Prob. 52. What two numbers are to each other as m to n, and the difference of whose squares is equal to b?

Ans. $\frac{m\sqrt{b}}{\sqrt{m^2-n^2}}$, $\frac{n\sqrt{b}}{\sqrt{m^2-n^2}}$.

Prob. 53. There are three pieces of cloth, whose lengths are in the ratio of 3, 4, and 5; and 12 yards being cut off from each, the whole quantity is diminished in the ratio of 7 to 4. What was the length of each piece at first?

Ans. 21, 28, and 35 yards.

Prob. 54. There are two boys, the difference of whose ages is to their sum as 1 to 3; and their sum is to their product as 3 to 8. What are their ages?

Ans. 4 and 8.

Prob. 55. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished twice as many men as there were companies in the regiment; but these being found insufficient, each company furnished five more men, when their number was found to be increased in the ratio of 5 to 4. How many companies were there in the regiment? *Ans.* 10.

Prob. 56. A certain sum of money, being put out at interest for 4 months, amounts to $1275. The same sum, put out at the same rate for 20 months, amounts to $1375. Required the sum and the rate per cent.
Ans. $1250, at 6 per cent.

Prob. 57. Divide the number 56 into two such parts that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{5}{3}$ to $\frac{3}{5}$. *Ans.* 35 and 21.

Prob. 58. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient will be 4; but if the digits be inverted, and that number be divided by a number greater by 4 than the difference of the digits, the quotient will be 9. Required the number. *Ans.* 36.

Prob. 59. There is a fraction whose numerator being doubled, and the denominator diminished by 2, the value becomes $\frac{5}{3}$; but if the denominator be tripled, and the numerator increased by 3, its value becomes $\frac{1}{3}$. Required the fraction. *Ans.* $\frac{5}{8}$.

Prob. 60. A father divided his estate among his five children, giving the first four $26,000; the last four,

\$30,000; the last three with the first, \$29,000; the first three with the last, \$27,000; and the first two with the last two, \$28,000. What was the share of each?

Ans. The first, \$5000; the second, \$6000; the third, \$7000; the fourth, \$8000; and the fifth, \$9000.

Prob. 61. There is an island 15 miles in circumference, and four men start together to travel the same way about it. A goes 4 miles per hour, B 5 miles, C 6 miles, and D 7 miles. How far will each travel before they all meet again at the spot from which they started?

Ans. A goes 60 miles; B 75 miles; C 90 miles; and D 105 miles.

Prob. 62. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits; and if 36 be added to it, the digits will be inverted. What is the number?

Ans. 48.

Prob. 63. A general ranging his army in the form of a solid square, finds he has 119 men to spare; but increasing each side by one man, he wants 100 men to fill up the square. How many soldiers had he?

Ans. 12,000.

Prob. 64. A square court-yard has a rectangular gravel-walk round it. The side of the court is 2 yards more than 6 times the breadth of the gravel-walk, and the number of square yards in the walk exceed the number of yards in the periphery of the court by 196. Required the breadth of the gravel-walk.

Ans. 3 yards.

Prob. 65. A and B set out from two towns, which were distant 204 miles, and traveled the direct road till they met. A went 8 miles per hour, and the number of hours at the end of which they met was greater by 3 than the number of miles which B went in an hour. How many miles did each go?

Ans. A 96 miles, and B 108 miles.

Prob. 66. A farmer bought a certain number of sheep for \$60; and if he had bought 10 more for the same money, they would have cost him one dollar a piece less. What was the number of sheep?

Ans. 20.

Prob. 67. A merchant sold a quantity of lace for \$56, and gained as much per cent. as the lace cost him. What was the cost of the lace?

Ans. \$40.

Prob. 68. A merchant sold a quantity of goods for a dollars, and gained as much per cent. as the goods cost him. What was the cost of the goods?

Ans. $\sqrt{2500+100a}-50$.

Prob. 69. Two merchants, A and B, enter into partnership with a joint stock of \$12,000. A's capital was employed 5 months, and B's 8 months. When the stock and gain were divided, A received \$7150, and B \$6120. What was each man's stock?

Ans. A's was \$6600, and B's \$5400.

Prob. 70. A merchant bought a number of pieces of cloth for \$225, which he sold again at \$16 by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces?

Ans. 15.

Prob. 71. A merchant bought a number of pieces of cloth for a dollars, which he sold again at m dollars by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces?

Ans. $\dfrac{a \pm \sqrt{4am+a^2}}{2m}$.

Prob. 72. A grocer sold 16 pounds of mace and 20 pounds of cassia for \$24; but he sold 10 pounds more of cassia for \$6 than he did of mace for \$5. What was the price of a pound of each?

Ans. Mace one dollar, and cassia forty cents.

Prob. 73. Two gentlemen, A and B, hired a pasture, into which A put two horses, and B as many as cost him \$4 a week. Afterward B put in two additional horses, and found that he must pay \$4.50 a week. At what rate was the pasture hired?

Ans. \$6 a week.

Prob. 74. What number is that to which, if 1, 4, and 10 be severally added, the first sum shall be to the second as the second to the third? *Ans.* 2.

Prob. 75. It is required to find three numbers such that the product of the first and second may be 24, the product of the first and third 32, and the sum of the squares of the second and third 100.

Ans. 4, 6, and 8.

Prob. 76. What are the values of x in the equation $3x^2-3x+6=5\frac{1}{8}$? *Ans.* $x=\frac{2}{8}$ or $\frac{1}{8}$.

Prob. 77. What are the values of x in the equation $x^2-x+3=45$? *Ans.* $x=7$ or -6.

Prob. 78. What are the values of x in the equation $5x^2-4x+3=159$? *Ans.* $x=6$ or $-\dfrac{26}{5}$.

Prob. 79. What are the values of x in the equation $3x-\frac{1121-4x}{x}=2$?

Ans. $x=19$ or $-\frac{59}{3}$.

Prob. 80. What are the values of x in the equation $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-2}{6}$?

Ans. $x=6$ or $\frac{1}{2}$.

Prob. 81. What are the values of x in the equation $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$?

Ans. $x=4$ or -1.

Prob. 82. What are the values ot x in the equation $\frac{x}{7-x}=\frac{7-x}{x}+2\frac{9}{10}$?

Ans. $x=5$ or 2.

Prob. 83. What are the values of x in the equation $\frac{x+12}{x}+\frac{x}{x+12}=5\frac{8}{15}$?

Ans. $x=3$ or -15.

Prob. 84. What are the values of x in the equation $16-\frac{2x^2}{3}=\frac{4x}{5}+7\frac{3}{5}$?

Ans. $x=3$ or $-\frac{21}{5}$.

Prob. 85. What are the values of x in the equation $\frac{16}{x}-\frac{100-9x}{4x^2}=3$?

Ans. $x=4$ or $\frac{25}{12}$.

Prob. 86. What are the values of x in the equation

$$3x-\frac{169-3x}{x}=29?$$

Ans. $x=13$ or $-\frac{13}{3}$.

Prob. 87. What are the values of x in the equation

$$\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}?$$

Ans. $x=3$ or $\frac{21}{11}$.

Prob. 88. What are the values of x in the equation

$$\frac{3x-4}{x-4}+1=10-\frac{x-2}{2}?$$

Ans. $x=12$ or 6.

Prob. 89. What are the values of x in the equation

$$\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}?$$

Ans. $x=12$ or 36.

Prob. 90. What are the values of x in the equation

$$\frac{90}{x}-\frac{27}{x+2}=\frac{90}{x+1}?$$

Ans. $x=4$ or $-\frac{5}{3}$.

Prob. 91. What are the values of x in the equation

$$\frac{4x^2+7x}{19}+\frac{5x-x^2}{3+x}=\frac{4x^2}{9}?$$

Ans. $x=3$ or $-\frac{87}{10}$.

Prob. 92. What are the values of x in the equation

$$3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}?$$

Ans. $x=4$ or $\frac{23}{2}$.

Prob 93. What are the values of x in the equation $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}$?

Ans. $x=2$ or $\frac{58}{13}$.

Prob. 94. What are the values of x in the equation $\frac{4}{2x+3}+\frac{3x+6}{5x+18}=\frac{3x+5}{5x}$?

Ans. $x=6$ or $-\frac{45}{2}$.

Prob. 95. What are the values of x in the equation $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$?

Ans. $x=3$ or $\frac{26}{11}$.

Prob. 96. What are the values of x in the equation $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$?

Ans. $x=1$ or -28.

Prob. 97. What are the values of x in the equation $\frac{x}{5+x}+\frac{7}{6-4x}=\frac{11x}{11x-8}$?

Ans. $x=1$ or $-\frac{40}{47}$.

Prob. 98. What are the values of x in the equation $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{8x}$?

Ans. $x=4$ or $-\frac{29}{9}$.

Prob. 99. What are the values of x in the equation

$$\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8?$$

Ans. $x=4$ or $-\frac{14}{3}$.

Prob. 100. What are the values of x in the equation

$$\frac{8}{9+5x}+\frac{8x-17}{2+4x}=\frac{4x+3}{2x+12}?$$

Ans. $x=3$ or $-\frac{285}{137}$.

THE END.